Portfolio Optimization with S+NuOpt: from Mean-Variance to Alternative Risk and Return Measures

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+ S+ optimization tools
+ Quadratic programming:
  • Cashflow matching liabilities
+ Portfolio Optimization with NuOpt
  • Mean-Variance approach
  • Expected Shortfall as a risk measure
  • Expected Growth Rate as a return measure
+ Q&A
+ **S+ optimization tools**
+ Quadratic programming:
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+ **Q&A**
SPLUS:
+ nlminb()
  • box-constrained NLP optimization

NuOPT:
+ solveQP()
  • General purpose quadratic/ linear program solver
+ SIMPLE
  • Built-in model definition language
S+NuOpt

+ Premier Toolkit for nonlinear optimization
+ Fully integrated with S-PLUS, with powerful statistics and graphics
+ Large-scale optimization problem solving, including:

• Linear programming
• Mixed integer programming
• Quadratic programming
• Unconstrained nonlinear optimization
• Multi-objective programming
ý **NETLIB problems.** There are many linear programming test problems in NETLIB (http://www.netlib.org/lp/index.html). NuOpt can solve 93 out of 95 problems without any parameter tuning.

ý **Hock and Schittkowski problems (Hock, 1980).** This test set includes some difficult highly nonlinear problems making it suitable for testing optimization algorithms. NuOpt can solve all 115 problems without any parameter tuning.

ý **CUTE test set.** (http://www.dci.clrc.ac.uk/Activity.asp?CUTE) contains about 800 problems which consist of linear and nonlinear, small to large problems. NuOpt was able to solve 150 out of 164 fairly large problems (up to 20000 variables and 10000 constraints). Of these, 18 problems need parameter tunings.
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**SIMPLE**

+System for Interactive Modeling in a Programming Language Environment

- variables (values to be determined through optimization)
- parameters (treated as constants in a model)
- expressions (mathematical expressions involving variables)
- constraints (mathematical relations that the solution of an optimization problem are required to obey)

+Procedure

- Create a function with SIMPLE commands that defines the optimization problem
- Call the NuOPT System() function
- Call the NuOPT solve() function

**S-Plus script**

```r
MaxDD.model <- function(rmat, rbar, rmin)
{
  -
  w <- Variable(index=i)
  rbar <- Parameter(rbar, index=i)
  rmin <- Parameter(rmin, changeable=H)
  -
  risk <- Objective(type="minimize")
  risk <- maxdd
  -
  sum{rbar[i, w[i], i] >= 8
  sum{w[i], i} == 1
}

MaxDD.system <- System(MaxDD.model, 
  rmat, rbar, rmin=x)
solution <- solve(MaxDD.system)
w <- solution$variables$w$current
```

Optimization function
+ S+ optimization tools
+ **Quadratic programming:**
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+ Q&A
“Easy” Optimization Tasks

+ Linear Programming (LP)
  • Objective function and constraints linear

+ Quadratic Programming (QP):
  • Objective function quadratic, constraints linear

+ Convex Programming (CP):
  • Objective function convex, constraints convex

Sample built-in S+NuOpt functions:

+ `solveQP (objQ, objL, A, cLB, cUB, bLB, bUB, x0, isint, type=minimize)`
+ `portfolioQPCov (rcov, averet, rmin)`
+ `portfolioQPSparse (rmat, averat, rmin)`
Cashflow matching liabilities

• The Problem:
  + We aim at finding the optimal portfolio of bond positions, whose associated cash flows replicate a set of liability payments.

• Relevant Objects:

\[ B = \begin{pmatrix} B_1 & B_2 & \cdots & B_K \end{pmatrix}^T \quad \leftarrow \text{vector of bond prices} \]

\[ l = \begin{pmatrix} l_1 & l_2 & \cdots & l_n \end{pmatrix}^T \quad \leftarrow \text{vector of liability cash flows} \]

\[ n = \begin{pmatrix} n_1 & n_2 & \cdots & n_K \end{pmatrix}^T \quad \leftarrow \text{vector of bond positions (amount invested)} \]

\[ C = \begin{bmatrix} C_{1,1} & \cdots & \cdots & C_{1,n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ C_{k,1} & \cdots & \cdots & C_{k,n} \end{bmatrix} \quad \leftarrow \text{matrix of bond cash flows} \]

See:
Scherer, B., 2002, Portfolio Construction and Risk Budgeting (Risk Waters Group)
Carrying cashflows

• Given a set \( n \) of bond holdings, one can calculate their deviation \( c \) from the required liability payments \( l \):

\[
c = C' \times n - l,
\]

where \( c \) is a vector of cash flow deviations –one per time step.

• If we can reinvest these cash flows at rates given by a vector \( r \), we can carry cash balances forward according to:

\[
ac(t+1) = c(t+1) + c(t) [1+r(t)]
\]

• Note that carrying excess cash flows forward always allows perfect replication –but it can be expensive.
  • If, across the maturity structure, some bonds are missing and we do not earn the forward rates on the excess cash carried forward, we have to buy big amounts of the remaining bonds to fill the cash flow gaps.

• Replication costs can be written as:

\[
\sum_{k=1}^{K} n_k B_k
\]
We can now express the total problem as minimizing replication costs under replication and non-negativity constraints:

\[
\begin{align*}
\min_{n, ac} & \quad B^T \cdot n \\
\text{subject to} & \quad C^T \cdot n + R \cdot ac = 1 \\
& \quad n \geq 0, ac \geq 0
\end{align*}
\]

SPLUS code with NuOpt:

```R
R <- diag(c(rep(-1, n))); for(i in 1:(n-1)){R[i+1,i]<-1+r[i]}
A <- cbind(CF, R)
obj<-c(matrix(rbind(matrix(B), matrix(rep(0,n)))))
x<-solveQP(obj,A,cLO=L,cUP=L,bLO=c(rep(0,length(B)),rep(0,n)),type=minimize)
```
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A portfolio is determined by the fractions $r_i$ of one’s capital $W(t)$ invested on each available asset $i$. Investors want to maximize the return of their portfolio of assets

$$R_p = \sum_i w_i R_i, \text{ where } R_i(t) = \frac{p_i(t) - p_i(t-1)}{p_i(t-1)}$$

Prices can be modeled as stochastic variables;

Average return of asset $i$: $a_i = \langle R_i \rangle$

Volatility of asset $i$: $\sigma_i = \sqrt{\langle (R_i - a_i)^2 \rangle}$

Portfolio variance: $\sigma_p^2 = \sum_i \sum_j w_j w_i \sigma_{ij}$
Main assumptions:
• Investors are risk averse
• Quadratic utility function
Volatility measures risk!

Mean-Variance approach:
Minimize the standard deviation $\sigma_p$ of a portfolio for any fixed value $\alpha_p$ of its expected return.
Existence of a risk-free asset (assume 0 interest).
Possible constraints: $r_i > 0$, $\Sigma r_i = 1$
Quadratic Portfolio Optimization

+ Markowitz optimization problem:

\[
\begin{align*}
\min & \quad \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\
\text{s.t.} & \quad \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} \\
& \quad \sum_{i=1}^{n} w_i = 1
\end{align*}
\]
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A better measure of risk

• **Std deviation:**
  „Quick and dirty“ risk measure.
  • Textbook risk measure
  • Violates monotonicity
  • Only Normal distributions are fully described

• **Value at Risk:**
  The amount which losses will not exceed within a given confidence interval.
  • Industry standard methodology
  • Violates subadditivity
  • To be used with elliptical distributions only

• **Expected Shortfall:**
  Average value of losses beyond VaR.
  • Coherent risk measure
  • Requires larger sample size
  • Satisfactory with (almost) any distribution
Expected Shortfall is the average of losses that exceed VaR.
Expected Shortfall Portfolio Optimization

Minimize:

$$R_{CVaR}(w, \beta) = R_{VaR} + \frac{1}{S} \sum_{s=1}^{S} \max(R_{VaR} - w'R_s, 0) \cdot \sum_{s=1}^{S} d_s$$

Average excess loss

Subject to:

$$d_s \geq R_{VaR} - w'R_s$$

$$d_s \geq 0$$

$$w'R \geq R_{min}$$

$$w'1 = 1$$

“Moderately Simple” Optimization Tasks

+ Mixed Integer Linear Programming (MILP)
  - Integer Constraints + LP

+ Mixed Integer Quadratic Programming (MIQP)
  - Integer Constraints + QP

Example: Markowitz with cardinality constraints

\[
\begin{aligned}
\min \quad & \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\
\text{subject to} \quad & \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} \\
& \sum_{i=1}^{n} w_i = 1 \\
& \delta_i w_i^{\min} \leq w_i \leq \delta_i w_i^{\max} \\
& \delta_i \in \{0, 1\} \\
& \sum_{i=1}^{n} \delta_i = \# \text{ of assets}
\end{aligned}
\]
Non-Linear Programming (NLP)

- Objective function and constraints nonlinear and not convex

No guarantee that the global optimum can be found.

An example follows...
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A better measure of return

• SHANNON, 1948, Bell Labs. Birth of Information Theory

• KELLY, 1956, Bell Labs (Bellman 57, Latané 59, Breiman 61)
  + Want to maximize your long-term investment returns while minimizing the risk of total ruin? Maximize the expected value of the logarithm of the capital (which is additive in repeated bets and to which the law of large numbers applies)

• CHAMBERS, BECKER, WILKS et al., 1976, Bell Labs
  + First Working version of S
Consider a game of outcome
\[ w(t+1) = A(t) w(t), \quad \text{where} \quad A = \begin{cases} 2 & \text{with prob. } p \\ 0 & \text{with prob. } q \end{cases} \]
with \( p > 0.5 \)

A gambler has a capital \( W(t) \). What percentage of it should he invest in a repeated game?

The value \( f = 1 \) maximizes the expected value of his capital, which would grow exponentially:

\[ \langle W(N) \rangle = (2p)^N W(0) \]

But \( W(N) \) goes to zero for large \( N \) with prob. 1!
Exponential rate of growth of the gambler's capital:

\[ G = \lim_{N \to \infty} \frac{1}{N} \log \frac{W(N)}{W(0)} \]

\[ W(N) = (1+f)^{N-L}(1-f)^L W(0) \]

Kelly’s solution: maximize \( G \) with respect to \( f \):

\[ \max_f (G) = 1 + p \log p + q \log q \Rightarrow f = 2p - 1 \]

In a nonterminating game, the gambler using this value of \( f \) will eventually get (and stay) ahead.
Assets’ prices follow (uncorrelated) geometrical Brownian motions:

\[ p_i(t+1) = e^{\eta_i(t)} p_i(t), \]

with \( \langle \eta_i \rangle = m_i \) and \( \langle (\eta_i - m_i)^2 \rangle = D_i \). Assume \( W(0)=1 \)

Wealth after 1 time step:

\[ W_1 = 1 + \sum_{i=1}^{N} r_i (e^{\eta_i} - 1) \]

The average return is

\[ \alpha_i = \exp[m_i + D_i/2] - 1 \]

and its variance

\[ \sigma_i^2 = (\exp[D_i] - 1) \exp[2m_i + D_i] \]
Following Kelly’s prescription, one can try and maximize the quantity

\[ v_{\text{typ}} = \langle \ln \left[ 1 + \sum_{i=1}^{n} r_i (e^{n_i} - 1) \right] \rangle \]

The naïve approximation \( \log(1+x) = x - x^2/2 \) is not realistic. Better to work with \( \partial_{r_i} v_{\text{typ}} \)

Main approximation for \( D << 1 \):

\[
\int_{-\infty}^{\infty} g(\eta) f(\eta) \, d\eta \approx g(m) + \frac{D^2}{2} g^{(2)}(m)
\]

Thus

\[
\left\langle \frac{e^{n_i} - 1}{W_1} \right\rangle \approx m_i + D_i (1 - 2r_i)
\]

Number of assets in portfolio

$N = 10^3$ assets, $m_i$ uniform in $[x-L, x+L]$, $x = -0.05$

$$M_T \rightarrow 1 + D \sqrt{\frac{N}{L}}$$
Mean-Variance vs. Median-Fluctuations

**Mean-Variance**

Fix a value $\mu_p$ of $<W>$ and minimize $\sigma_p$ with respect to the $r_i$.

**Median-Fluctuations**

Fix a value $v_p$ of $<\log W>$ and minimize $<|\log W - v_p|>$ with respect to the $r_i$.

Plot both results in the plane $(\mu_p, \sigma_p)$. 
\( \mu_P \) vs. \( \sigma_P \)

- **single assets**
- **unchostrained**
- **positive**
Conclusions

- Expected Shortfall is a coherent risk measure, which can be used in portfolio management whenever there is enough data.

- Kelly’s Criterion has some advantages over MV optimization:
  - it is better in the long run, it does not require utility functions nor the existence of the moments of the return distribution.
  - Median-fluctuation procedures can be used to trace efficient frontiers.

- Optimal portfolios condensate whenever there is a “dominating asset” (or group of assets).

- NuOpt can solve efficiently advanced optimization problems.

