Wavelet Analysis on Stochastic Time Series
A visual introduction with an examination of long term financial time series

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Contents

1 Introduction ............................................. 5
   1.1 Motivation - Musical Notation ..................... 5
   1.2 Outline .............................................. 6
   1.3 Technical Notes ...................................... 6

2 Background ............................................. 7
   2.1 Introduction ......................................... 7
   2.2 Wavelet Functions .................................... 8
      2.2.1 Morlet Wavelet (time domain) .................. 9
      2.2.2 Morlet Wavelet (frequency domain) .......... 9
   2.3 Wavelet Transformation ............................... 11
      2.3.1 Choice of Scales ................................ 12
      2.3.2 Wavelet Scale and Fourier Frequency ......... 13
      2.3.3 Cone of Influence ................................ 13
   2.4 Power Spectrum ...................................... 14
      2.4.1 Stochastic time series ......................... 14
      2.4.2 Fourier AR(1) spectrum ....................... 15
      2.4.3 Wavelet AR(1) spectrum ....................... 16
      2.4.4 Contour Plot ................................... 16
      2.4.5 Significance levels ............................. 17
      2.4.6 Reconstruction ................................. 17

3 Artificial Time Series ................................. 19
   3.1 Simulation of Time Series ............................ 19
      3.1.1 White Noise .................................... 21
      3.1.2 Brown Noise .................................... 21
      3.1.3 Including Features .............................. 22
   3.2 Featured Time Series ................................ 25
      3.2.1 About Linear Trends ............................ 25
      3.2.2 Reality Check .................................. 26
      3.2.3 Brown Noise Modifications .................... 26
      3.2.4 White Noise Modifications ..................... 26
   3.3 Examples ........................................... 28
      3.3.1 Jumps and Steps ................................ 29
      3.3.2 Oscillations .................................... 31
      3.3.3 Volatility Clusters ............................. 33

4 Real World Time Series ............................... 35
   4.1 Introduction ......................................... 35
      4.1.1 Data Source ..................................... 35
      4.1.2 The Trader Model ............................... 35
1 Introduction

Wavelet analysis is a method to decompose a time series into time-frequency space. This view offers interesting insights into the dominant modes of a time series and how those modes vary over time.

The aim of this paper is to give an introduction into the field of wavelets by using a rather visual approach. Two types of financial time series will be examined. First, artificial time series that were generated by including trends and disturbances into a random process and second, real world long term financial time series that date back until the time before the great depression. For all those time series wavelet spectras were calculated and visualized. To understand what kind of information a wavelet spectrum is providing, it very much helps to visualize those spectras in an appealing way.

Of course the wavelet spectrum can only be fully understood by examining the theory behind it. Therefore a description on how the wavelet spectras in this paper were calculated will be given; along with a description on how we can interpret the values of such a spectrum from a mathematical point of view. The theory may only cover a small part of the whole field of wavelet theory. But is should be enough to get a solid base on which further and more complex work can be developed, such as strategies for financial trading.

1.1 Motivation - Musical Notation

To illustrate the idea about wavelets I would like to make a little excursion into the field of musical notation; inspired by the standard treatise ”Ten lectures on wavelets” of Daubechies [3].

![Figure 1: Prelude Opus 28 Number 7 by Frédéric Chopin](image)

Musical notation shows us which frequencies are played at which point in time, how long those frequencies are played and what volume (amplitude) they have. It would be desirable if we could describe (stochastic) time series in the same way. Wavelet analysis is a method for such an approach.

Fourier transformation is such a method as well. But unlike wavelet transformation it doesn’t deliver an indication about the points in time where a certain frequency was played. In the context of music the result of a fourier transformation can be described as a time average of the song. This would only be an appropriate description of the song if every instrument would play the same tone for the whole duration of the song.
Speaking about time series again, the analogue of the rather boring song, where the instruments don’t change their tone, would be a stationary signal, in the sense that the frequency of that signal remains constant over time. Since financial time series often show very localized behavior, the wavelet approach is much better suited to handle them than the fourier approach.

Some of the readers may think about the windowed fourier transformation which would indeed offer a time indication as well. But unlike the wavelet transformation it imposes a ”response interval”, which means that frequencies above (depending on the size of the window) and under (depending on the size of the time step) a certain threshold cannot be detected satisfyingly.

1.2 Outline

Section 2 presents the theory of creating and visualizing a wavelet spectrum and provides a mathematical description of the values of such a spectrum. Section 3 introduces into the topic of simulating stochastic time series and presents wavelet spectras of such artificial time series. On the other hand section 4 presents wavelet spectras of real world financial time series. Finally in section 5 some possible applications of how wavelets could explicitly be used in statistics and finance are described.

1.3 Technical Notes

This paper is an example of literate programming. Which means that the source code of this paper is a mixture between R and \LaTeX. By compiling the source code all the calculations and plots are recalculated or created again and included into the resulting document. This document was created on a Windows 7 x64 platform using the Sweave library and version 2.14.2 of R.

Especially the analysis in section 3 and section 4 are done in that spirit. In section 1 and section 2 some figures were created outside R since those only serve to explain the theory and do not often have to be recreated.

This paper is actually based on a presentation initially hold on the 2011-08-13 on ETH Zürich which may contain some additional information.
2 Background

The calculations and plots of the wavelet spectras in this paper were done by using the \texttt{dplR} library (version 1.5.3). The implementation of the wavelet spectrum calculation within that library is based on the "Practical guide to wavelet analysis" by Torrence and Compo [6]. Therefore the content of this section follows that paper very closely. However some of the formulas were slightly modified or additionally added to make sure that this background section is as close as possible to the actual implementation.

2.1 Introduction

To perform a wavelet analysis we basically only need a time series and a wavelet function.

\begin{figure}[h]
\centering
\begin{minipage}[b]{0.4\textwidth}
\includegraphics[width=\textwidth]{figure2}
\caption{Stochastic time series having an oscillatory signal in it.}
\end{minipage}\hspace{1em}
\begin{minipage}[b]{0.4\textwidth}
\includegraphics[width=\textwidth]{figure3}
\caption{The real part of the morlet wavelet function.}
\end{minipage}
\end{figure}

The focus in this paper is on stochastic time series that have possible oscillations, peaks or discontinuities in them. In their original form, time series are often not suited to perform a wavelet analysis. More on that topic can be found in section 2.4.1 and section 3.

All the wavelet transformations in this paper were done by using the morlet wavelet. In section 2.2 the general requirements on wavelet functions are presented and the morlet wavelet is described in detail.

Once a suitable time series is available and a wavelet function is chosen, the wavelet coefficients (the wavelet spectrum) can be determined by calculating the scalar product between the time series and various altered versions of the wavelet function. The calculation of the wavelet coefficients will be treated in section 2.3.

In section 2.4 a mathematical description of the wavelet spectrum (more precisely the power spectrum) will be presented by examining the special case of AR(1) processes.
2.2 Wavelet Functions

There are various different wavelet functions available to perform a wavelet analysis. In order to choose one, it has to fulfill certain properties. Essentially it is enough to make sure that the wavelet function $\Psi$ is absolutely and square integrable, which means that it belongs to $L^1 \cap L^2$ and that $\Psi$ has zero mean, which means that $\int_{-\infty}^{\infty} \Psi(t) dt = 0$. By choosing a wavelet function it pays off to consider the following factors:

- **Orthogonal or Nonorthogonal**
  Using orthogonal wavelets, the number of convolutions (see section 2.3) at a certain scale is proportional to the width of the wavelet function at that scale. This gives the most compact representation of the time series under consideration.
  Using nonorthogonal wavelets, the number of convolutions at each scale is constant and equal to the number of observations. As opposed to orthogonal wavelets, this leads to correlations in the wavelet spectrum at adjacent times but is better suited to outline smooth, continuous variations in the wavelet amplitude.

- **Complex or Real**
  Complex wavelet functions return information about amplitude and phase and are therefore better adapted to capture oscillatory behavior. Real wavelet functions only return information about amplitude and are therefore better suited to find peaks or discontinuities.

- **Width**
  The resolution of a wavelet function is determined by the balance between the width in real space and in fourier space. A narrow function in time will have good time resolution but poor frequency resolution. While for a broad function in time it is the other way round.

- **Shape**
  The wavelet function should reflect the type of features present in the time series. If one is only interested in the power spectrum then the choice of a wavelet function is not that critical since they will lead to the same qualitative results.

All the wavelet transformations in this paper were performed by using the morlet wavelet. The morlet function is nonorthogonal, complex and seems to offer a good trade-off between detecting oscillations and peaks or discontinuities. In the following the morlet wavelet function will be described in the time and frequency domain.
2.2.1 Morlet Wavelet (time domain)

The morlet wavelet, a plane wave modulated by a gaussian, is defined as follows:

\[ \Psi_0(\eta) = \pi^{-1/4} e^{i\omega_0 \eta} e^{-\eta^2/2} \]

where \( \eta \) is a nondimensional "time" parameter (see also figure 4) and \( \omega_0 \) is equal to 6 in order to guarantee the admissibility of the wavelet function. \( \Psi_0(\eta) \) is complex, nonorthogonal and normalized to have unit energy.

To calculate the wavelet coefficients (section 2.3) we will use scaled and translated versions of the above mother wavelet in a discrete form:

\[ \Psi\left[\frac{(n' - n) \delta t}{s}\right] = \left(\frac{\delta t}{s}\right)^{1/2} \Psi_0\left[\frac{(n' - n) \delta t}{s}\right] = \Psi_{n,s} \]

The value of \( n \) describes where the highest peak of the wavelet function is situated. The value of \( s \) (a multiple of \( \delta t \)) is a scaling factor. The factor in front of \( \Psi_0 \) assures that \( \Psi \) is normalized to have unit energy.

Figure 4 shows \( \Psi_0(\eta) \) using \( \delta t = 0.01, s = 100\delta t = 1 \) and \( n = 0 \) which means that \( \eta = (n' \delta t)/s = t/s \). If we would take a \( n > 0 \) the wavelet function would move to the right and the bigger we choose \( s \), the bigger the width of the wavelet function. The width can be expressed through \( \lambda_s \) (section 2.3.2) or \( \tau_s \) (section 2.3.3).

2.2.2 Morlet Wavelet (frequency domain)

The fourier transform of \( \Psi_0(t/s) \) is given as follows:

\[ \hat{\Psi}_0(s\omega) = \pi^{-1/4} H(\omega)e^{-(s\omega - \omega_0)/2} \]

where \( H(\omega) \) is the heaviside step function (\( H(\omega) = 1 \) if \( \omega > 1 \), \( H(\omega) = 0 \) otherwise) and \( \hat{\Psi}_0(1 \cdot \omega) \) is normalized to have unit energy. The variables \( \omega_0 = 6 \) and \( s \), the scaling factor, are defined as in the time domain (see section 2.2.1).

The discrete and scaled version of the morlet wavelet in fourier space is formulated as follows:

\[ \hat{\Psi}(s\omega_k) = \left(\frac{2\pi s}{\delta t}\right)^{1/2} \hat{\Psi}_0(s\omega_k) = \hat{\Psi}_s \quad \text{where} \quad \omega_k = \frac{2\pi k}{N\delta t} \]

Again the factor in front of \( \hat{\Psi}_0 \) assures that \( \hat{\Psi} \) is normalized to have unit energy. In discrete notation this is:

\[ \sum_{k=0}^{N-1} |\hat{\Psi}(s\omega_k)|^2 = N \]

The reason why we even need the fourier transform of the morlet wavelet is that it is considerably faster to calculate the wavelet coefficients in fourier space (section 2.3). Also it helps to understand the nature of the fourier coefficients (section 2.4) and how the wavelet scale \( s \) is related to a fourier frequency \( \lambda_s \) (section 2.3.2).

Figure 5 shows the fourier transform of the wavelet function that is shown in figure 4 (see section 2.2.1).
Figure 4: The morlet wavelet function in the time domain. The solid line shows the real part and the dashed line the imaginary part.

Figure 5: The morlet wavelet function in the frequency domain.
2.3 Wavelet Transformation

Assume that we have a discrete time series \( r_n \) with equal time spacing \( \delta_t \) and \( n = 0, 1, ..., N - 1 \). The continuous wavelet transform of \( r_n \) is then defined as:

\[
W_n(s) = \sum_{n'=0}^{N-1} r_{n'} \cdot \Psi^* \left[ \frac{(n' - n)\delta_t}{s} \right]
\]

where \( \Psi_{n,s} \) can be any wavelet function (see section 2.2) and \((*)\) indicates the complex conjugate. For a fixed scale \( s \) this expression is also called a convolution. If we additionally fix \( n \), the formula above can be described as a scalar product of \( r_n \) and \( \Psi_{n,s} \).

![Figure 6: Wavelet spectrum of white noise.](image.png)

The \( W_n(s) \) are the wavelet coefficients. They are complex and to visualize them we will use the square of their absolute value, denoted as \( |W_n(s)|^2 \). Those squared values are also called the wavelet (or power) spectrum (see section 2.4). An example of such a power spectrum can be seen in the lower part of figure 6.

If we think of a specific \( |W_n(s)|^2 \) as a scalar product again, a possible interpretation is that the bigger the scalar product, the more \( r_n \) and \( \Psi_{n,s} \) look alike. A high value would result in a dark red point in figure 6 and a low value in a bright yellow point. We can now fix a point in time \( n \) and calculate a value for a certain \( s \). This will then tell us how prominent a certain frequency \( \lambda_s \) (see section 2.3.2) is within the time series \( r_n \) at time \( n \).
A computationally more efficient way to calculate the wavelet coefficients is to do it in fourier space. By using the convolution theorem we get the following formula for the calculation of the wavelet coefficients:

\[ W_n(s) = \sum_{k=0}^{N-1} \hat{r}_k \hat{\Psi}^*(s\omega_k) e^{i\omega_k n\delta t} \]

where \( \hat{r}_k \) is the discrete fourier transform (DFT) of the original sequence \( r_n \):

\[ \hat{r}_k = \frac{1}{N} \sum_{n'=0}^{N-1} r_{n'} e^{-i\omega_k n' \delta t} \]

and \( k = 0 \ldots N - 1 \). The fourier coefficients \( \hat{r}_k \) can be calculated efficiently by using a fast fourier transformation (FFT) algorithm. The advantage of calculating \( W_n(s) \) in fourier space is that for a given \( s \) the wavelet coefficients of all \( n \) can be calculated simultaneously.

The implementation of the dplR library uses R’s internal \texttt{fft()} function to calculate the wavelet coefficients. Note that this function is actually performing a centered discrete fourier transformation (CDFT). The key idea is to define the \( \omega_k \) as follows:

\[ \omega_k = \begin{cases} \frac{2\pi k}{N\delta t} & 0 \leq k \leq \frac{N}{2} \\ \frac{2\pi (N-k)}{N\delta t} & \frac{N}{2} < k \leq N - 1 \end{cases} \]

### 2.3.1 Choice of Scales

The scales \( s \) are chosen as follows:

\[ s_j = s_0 2^{j\delta j} \quad \text{where} \quad j = 0, 1 \ldots J \]

The artificial time series in section 3 consist of \( N = 1000 \) samples and the wavelet spectrum was calculated by setting \( s_0 = \delta t, \delta t = 1 \) and \( \delta j = 1/4 \). This means that \( 1/\delta j - 1 = 3 \) additional scales between \( 2^j \) and \( 2^{j+1} \) were considered. The maximum value of \( j \) can be calculated as follows:

\[ J = \left\lfloor \frac{1}{\delta_0} \log_2 \left( \frac{N\delta t}{s_0} \right) \right\rfloor \]

where \( \lfloor ... \rfloor \) is the floor operation. For the artificial time series in section 3 this leads to a \( J \) of 39 and therefore the maximal considered scale is \( s_J = 861 \).
2.3.2 Wavelet Scale and Fourier Frequency

One might assume that the frequency of the wavelet function $\Psi_{n,s}$ is the inverse of the scale ($s^{-1}$). This would imply that the fourier transform of the wavelet function ($\hat{\Psi}_s$) has a peak at $s^{-1}$. As we can see in figure 5 this is approximately true for the morlet wavelet. For other wavelets this is usually not the case.

The exact formulation to calculate the fourier frequency for the morlet wavelet goes as follows:

$$\lambda_s = \frac{4\pi s}{\omega_0 + \sqrt{2 + \omega_0^2}} = c \cdot s$$

As introduced in section 2.2.1 we have $\omega_0 = 6$. It seems now reasonable that $c$ has a value close to 1 and indeed we get $\lambda_s = 1.03s$. Consult also figure 4 for an alternative visual interpretation of the fourier frequency.

2.3.3 Cone of Influence

The fact that finite-length time series are used, leads to the problem that the wavelet function is defined beyond the borders of the time series. This is visualized in the upper part of figure 6. Providing the original time series to the $\text{fft()}$ function, it would just assume that the time series is cyclic. Which is of course not the case.

A possible solution is to pad the time series with zeros. The amount of zeros that are added should bring the total length of the padded time series to the next higher power of two. The artificial time series in section 3 consist of $N = 1000$ samples, the length of the padded time series would therefore be $N' = 2048$.

This procedure will lead to edge effects which become important inside the so called cone of influence (COI). The COI is visualized in the lower part of figure 6 as the shaded area. Inside the COI the values of the wavelet coefficients $W_n(s)$ are reduced due to the zero padding. Note that for cyclic series there is no COI.

The COI is defined by the e-folding time $\tau_s$ which is chosen such that a discontinuity drops by a factor of $e^{-2}$ and ensures that edge effect are negligible beyond this point:

$$\tau_s = \sqrt{2s} = \frac{\sqrt{2}}{c} \lambda_s$$

For every point in time $n$, the COI can be calculated as follows:

$$y_n = \log_2 \left( \frac{c}{\sqrt{2}^n} \right)$$

where the logarithm is used since the wavelet spectrum has an exponentially growing $y$ axis. This is due to the scales that are always a power of 2.

The e-folding time $\tau_s$ is visualized in figure 6. By comparing the width of a significant area (w) in the wavelet power spectrum with this decorrelation time, one can distinguish between a spike in the data and a harmonic component at the equivalent fourier frequency. Consult also figure 4 for an alternative visual interpretation of the e-folding time.
2.4 Power Spectrum

Because the wavelet function $\Psi$ is complex, so are the wavelet coefficients $W_n(s)$. To visualize them, real numbers are needed. This leads to different possibilities on how to create the wavelet spectrum. Possibilities are to plot the real part $\Re(W_n(s))$, the imaginary part $\Im(W_n(s))$, the phase $\tan^{-1}(\Re(W_n(s))/\Im(W_n(s)))$ or the power spectrum $|W_n(s)|^2$. In this paper all the plots show the power spectrum, an example can be seen in the lower part of figure 6. More on how the visualization is done can be found in section 2.4.4.

2.4.1 Stochastic time series

This paper focuses on the case where the wavelet transformation is applied to stochastic time series. A widely used model to describe financial and other stochastic time series is the univariate lag-1 autoregressive (AR(1)) process which is defined as follows:

$$ r_n = m + \alpha r_{n-1} + \epsilon_n $$

where $m$ is a constant, $\alpha$ the lag-1 autocorrelation coefficient and the $\epsilon_n \sim N(0, \sigma^2_\epsilon)$ are normal distributed numbers. The mean $\mu$ and the variance $\sigma^2$ of such a process can be calculated as follows:

$$ \mu = \frac{m}{1-\alpha}, \quad \sigma^2 = \frac{\sigma^2_\epsilon}{1-\alpha^2} $$

and are constant over time.

If $m = 0$ and $\alpha = 0$ we are talking about white noise. Figure 6 shows the wavelet transformation of such a process. If $m = 0$ and $\alpha \to 1$ the AR(1) process converges into brownian noise (also known as red noise) which has infinite variance. A stationary process, in the sense that the mean and the variance of $r_n$ stay constant, is therefore only guaranteed if $|\alpha| < 1$.

Within the following sections the theoretical wavelet spectrum for AR(1) processes where $m = 0$ is derived. Not that an AR(1) process where the mean and therefore also $m$ is not zero can be brought into the desired form by calculating the mean $\mu$ of the process and subtracting it from the samples $r_n$.

Deriving the distribution of the wavelet coefficients $W_n(s)$ allows to define the theoretical expectation values $E[|W_n(s)|^2]$ of the spectrum. Moreover significance levels can be defined in order to find areas that are considerably above the expected values.
2.4.2 Fourier AR(1) spectrum

The theoretical power spectrum for white noise \((r_n = \epsilon_n)\) can be derived as follows:

\[
E[|\hat{r}_k|^2] = E \left[ \left| \frac{1}{N} \sum_{n=0}^{N-1} r_n e^{-i\omega_k n\delta t} \right|^2 \right] \cdot \frac{1}{N^2} \sum_{n=0}^{N-1} E \left[ |r_n|^2 \right] \left| e^{-i\omega_k n\delta t} \right|^2 =
\]

\[
= \frac{1}{N^2} \cdot N(\sigma^2 \cdot 1) = \frac{\sigma^2}{N}
\]

by applying that \(E[r_i r_j] = \text{cov}(r_i, r_j) = 0\) for \(i \neq j\). For \(0 < |\alpha| < 1\) the covariances are not zero. In that case, following Gilman et al. (1963), the expected discrete fourier power spectrum of an AR(1) process with zero mean \((m = 0)\) can be formulated as:

\[
E[|\hat{r}_k|^2] = P_k \frac{\sigma^2}{N}
\]

where

\[
P_k = \frac{1 - \alpha^2}{1 + \alpha^2 - 2\alpha \cos(\omega_k \delta t)}
\]

and \(k = 0 \ldots N/2\) is the frequency index. Both formulas state that for white noise the expected value of \(|\hat{r}_k|^2\) is \(\sigma^2 / N\) for all frequencies \(\omega_k\) (as defined in section 2.3). A closer look at \(P_k\) (see figure 7) shows that the difference between the expectation value for high frequencies and low frequencies increases with increasing \(\alpha\).

Figure 7: Examples of \(P_k\).
2.4.3 Wavelet AR(1) spectrum

Similar to section 2.4.2 we can now derive the theoretical power spectrum of an AR(1) process with zero mean ($m = 0$):

$$E[|W_n(s)|^2] = E \left[ \sum_{k=0}^{N-1} \hat{r}_k \hat{\Psi}^*(s\omega_k)e^{i\omega_k n\delta t} \right]^2 = \sum_{k=0}^{N-1} E \left[ |\hat{r}_k|^2 \right] |\hat{\Psi}(s\omega_k)|^2 |e^{i\omega_k n\delta t}|^2$$

$$= \frac{\sigma^2}{N} \sum_{k=0}^{N-1} P_k \cdot |\hat{\Psi}(s\omega_k)|^2 \cdot 1 = P_s \sigma^2$$

where

$$P_s = \frac{1}{N} \sum_{k=0}^{N-1} P_k \cdot |\hat{\Psi}(s\omega_k)|^2$$

and $E[\hat{r}_i \hat{r}_j] = \text{cov}(\hat{r}_i, \hat{r}_j) = 0$ for $i \neq j$ is assumed. It is not clear to me if this is true. Torrence et al. (1997) performed a monte carlo simulation which highly suggests that the formula indeed holds. They compared the theoretical value of $P_s$ to the average power spectrum of 100’000 simulated red noise processes which resulted in very high accordance.

Note that for white noise the expected value of $|W_n(s)|^2$ is $\sigma^2$ at every point in time $n$ and for every fourier frequency $\lambda_s$ (as defined in section 2.3.2).

2.4.4 Contour Plot

The values of the power spectrum $|W_n(s)|^2$ are visualized by using contour plots. An example can be seen in the lower part of figure 6. The current implementation of the dplR library plots the original values of $|W_n(s)|^2$.

An alternative would be to divide those value by $\sigma^2$ which would result in a spectrum relative to white noise or by $P_s \sigma^2$ which would result in a spectrum relative to an AR(1) process. If two time series can be described in good approximation as AR(1) processes this would make their contour plots better comparable and if $\alpha$ is close to one it would lower the effect of increasing power spectrum values in scale direction (see figure 7).

The current levels for the contour plots are computed by performing a quantile analysis using the original values of $|W_n(s)|^2$ for $p' \in [0, 0.1, ..., 0.9, 1]$. This makes sure that in every level the same amount of points are present. Alternatively those levels could be defined by the theoretical significance levels (see section 2.4.5).
2.4.5 Significance levels

In order to be able to define theoretical significance levels we need to know how the wavelet power spectrum $|W_n(s)|^2$ is distributed.

If the samples $r_n$ are normally distributed then the real and imaginary part of $\hat{r}_k$ are both normally distributed as well. The square of a normally distributed number is chi-square distributed with one degree of freedom, therefore $|\hat{r}_k|^2$ is chi-square distributed with two degrees of freedom.

Since the local wavelet power spectrum follows the mean fourier spectrum (see section 2.4.3), the distribution of $|W_n(s)|^2$ is the same as for $|\hat{r}_k|^2$. Therefore every value of the wavelet power spectrum is distributed as follows:

$$\frac{|W_n(s)|^2}{\sigma^2 P_s} \sim \frac{1}{2} \chi^2_k$$

where $E[\chi^2_k] = k$. Since the current implementation of the dp1R library doesn’t perform any scaling, the significance level $\rho$ for a certain probability $p$ is calculated as:

$$\rho = \frac{1}{2} \sigma^2 P_s \chi^2_k(p)$$

For a white noise process having $\sigma^2 = 1$ the value of $\rho$ for $p = 0.95$ is approximately 3. An example are the thick contour lines in the lower part of figure 6 which show the values within the spectrum that lie above the 95% significance level for a white noise process having $\sigma^2 = 0.0025$.

2.4.6 Reconstruction

For the sake of completeness it should just be mentioned here that it is possible to reconstruct the original time series $r_n$ using the wavelet coefficients $W_n(s)$. It has to be taken special care of the fact that the wavelet coefficients of a continuous wavelet transform are correlated in time and frequency. More on that topic can be found in the "Practical guide to wavelet analysis" by Torrence and Compo [6].
3 Artificial Time Series

This section focuses on artificial time series in order to get a qualitative feeling on how to interpret the outcome of a wavelet transformation. The dplR library (version 1.5.3) offers a function to calculate the wavelet coefficients $W_n(s)$ of a time series $r_n$ by using the morlet wavelet and a plot function to visualize the power spectrum $|W_n(s)|^2$ of that series (see also section 2.4).

The following figures were created by using a slightly modified version of the dplR wavelet plot function. Besides some minor changes on the layout the modified version provides the possibility to add a second time series to the upper part of the plot, having dates on the x-axis and choosing between pre-defined color schemes that are used to visualize the power spectrum in the lower part of the plot.

In the following it will be shown how the wavelet spectrum reacts on specific features (like oscillations or peaks) that are within the time series. How those time series were created will be explained in section 3.1. A possible classification of the features used to disturb the time series is outlined in section 3.2 and finally in section 3.3 examples of various disturbed time series are collected.

3.1 Simulation of Time Series

The basis of all the artificial time series within this paper is the white noise process described in section 3.1.1. In some approximation white noise can be used as a model for financial return series. It is no secret that real world financial time series often behave differently.

To simulate the unstationary behavior of financial time series in some extent, different features (section 3.1.3) (like oscillations, peaks or volatility clusters) were included into the process. This was done by either including a feature directly into the original white noise or by using that white noise to generate brown noise (section 3.1.2) first and include the feature there.

To get even closer to financial time series we could also use an AR(1) process that has an $\alpha > 0$ (see section 2.4.1). The simple white noise model was chosen since the theoretical wavelet spectrum of white noise can be well described and it is therefore a good base in order to better understand how the wavelet spectrum is affected by disturbances. Also it seems that the impact of a certain disturbance (within the time series) on the wavelet spectrum does not depend that much on the underlying process.

A wavelet analysis should only be done for AR(1) like processes as further explained in section 3.1.2. Therefore, after including a feature into a brown noise process, the process should be differentiated again before performing a wavelet analysis. Note that while white noise can be seen in some approximation as a model for financial returns, brown noise can be seen as an approximative model for financial logarithmic price series that have no trend.
Figure 8: White Noise.

Figure 9: Brown Noise.
3.1.1 White Noise

A white noise process can be described as follows:

$$r_n = \epsilon_n$$

where the $$\epsilon_n \sim N(0, \sigma^2_\epsilon)$$ are normal distributed numbers with zero mean. An example can be seen in figure 8. The upper part of the figure shows the white noise process $$r_n$$ while the lower part shows the power spectrum $$|W_n(s)|^2$$ of the process. The process was generated by producing $$N = 1000$$ normal distributed numbers having mean $$\mu = 0$$, variance $$\sigma^2_\epsilon = 0.0025$$ and $$\delta t = 1$$.

All the artificial time series shown in this paper were generated by using the white noise process shown in figure 8. As outlined in section 2.4.1 white noise is an AR(1) process where $$\alpha = 0$$ and $$m = 0$$. Setting $$\alpha > 0$$ would lead to the same qualitative results but implementing more complexity at the same time. Therefore white noise was chosen.

Referring to section 2.4.3 and section 2.4.5 the expectation value and the distribution of a white noise wavelet spectrum can be formulated as follows:

$$E[|W_n(s)|^2] = \sigma^2_\epsilon, \quad |W_n(s)|^2 \sim \frac{1}{2}\sigma^2_\epsilon \chi^2_2$$

This means that the values of the power spectrum in figure 8 are chi-square distributed (with two degrees of freedom) ($$\chi^2_2$$) around the variance of the process ($$\sigma^2_\epsilon$$). For the presentation mentioned in section 1.3 a monte carlo simulation was performed that visualizes that the values of the power spectrum indeed converge towards $$\sigma^2_\epsilon$$, except within the COI (see section 2.3.3) where the values are lowered due to the zero padding.

3.1.2 Brown Noise

A brown noise process can be created from a white noise process as follows:

$$B_n = B_{n-1} + r_n$$

where $$S_0 = 0$$. The brown noise process in figure 9 was created by using the white noise process that can be seen in figure 8.

Referring to section 2.4.1 this process could be described as an AR(1) process where $$\alpha = 1$$ and $$m = 0$$. This would imply that the variance $$\sigma^2$$ of this process tends to infinity and therefore also the expectation values of the wavelet spectrum $$E[|W_n(s)|^2]$$. But since we use finite length time series and numerical methods to perform a wavelet analysis, the values of the wavelet spectrum will be finite. Even if a process was created by setting $$\alpha = 1$$. In fact, every time series that is provided to the function that calculates the wavelet coefficients is examined for autocorrelation. This means that the $$\alpha$$ for the process is calculated numerically which will lead to an $$\alpha$$ that is close to one but not equal to one. Therefore, what we see in figure 9 is actually the spectrum of an AR(1) process that has an $$\alpha$$ which is close to one. We can see that the spectrum is growing strongly in scale direction. This is in accordance with the content of section 2.4.2 and section 2.4.3 (see also figure 7).
For the sake of completeness the expectation value and the distribution of an AR(1)
wavelet spectrum as derived in section 2.4.3 and section 2.4.5 should be reformulated
here:

$$E[|W_n(s)|^2] = P_s \sigma^2, \quad |W_n(s)|^2 \sim \frac{1}{2} \sigma^2 P_s \chi^2$$

Generally it is important to always make sure that the time series under consideration
has finite moments and can be described in some approximation as an AR(1) process.
This is not true for brown noise and that’s why we use differentiated brown noise to
perform a wavelet analysis:

$$r_n = B_n - B_{n-1}$$

For the same reasoning we don’t analyze financial price series directly but use the log-
returns:

$$r_t = \log \left( \frac{S_t}{S_{t-1}} \right) = \log(S_t) - \log(S_{t-1})$$

Note that a positive side effect of differentiating the time series is that the signal-noise
ratio improves. This is visualized in section 3.3.

### 3.1.3 Including Features

To visualize how the wavelet spectrum reacts on disturbances, different features were
added to the process. Depending on the feature it was either added to the white noise
process or the brown noise process. A linear trend can usually be seen within logarithmic
financial price series. Therefore that feature was added to the brown noise process. On
the other hand volatility clusters can usually be seen within financial returns series and
were therefore introduced directly into the white noise process.

The procedure can now be summarized as follows:

1. Create white noise ($r_n$).
2. If a feature should be simulated that can usually be seen in financial returns, use
   the white noise to implement the feature.
3. If a feature should be simulated that can usually be seen in logarithmic financial
   price series, create brown noise ($B_n$) and use it to implement the feature.
4. Modified white noise can be directly analyzed. Modified brown noise should be
differentiated before analyzing.

More details on which features are implemented in which process and on what can be
expected by such disturbances are given in section 3.2. Actual examples to illustrate the
content of section 3.2 are given in section 3.3.
Figure 10: Exponential brown noise with trend.

Figure 11: Brown noise with trend.

Figure 12: White noise from brown noise with trend.
Figure 13: SP500 price series.

Figure 14: SP500 logarithmic price series.

Figure 15: SP500 returns series.
3.2 Featured Time Series

In order to visualize how a certain feature within a time series impacts the wavelet spectrum, a base process is needed that can be disturbed. In section 3.1.3 we claimed that this can either be white or brown noise. One might think that plain brown noise is not a good model since financial time series usually show a linear trend. This will be examined in section 3.2.1. In section 3.2.2 the undisturbed white and brown noise processes will be compared with a real world financial price series.

Three features were picked to disturb a certain base process:

- Oscillations
- Jumps and Steps
- Volatility Clusters

In section 3.2.3 the features that were included into brown noise will be described. Those are the oscillations, jumps and steps. In section 3.2.4 the features that were included into white noise will be described. This are only the volatility clusters. Section 3.3 presents one example for every of those features. More examples are gathered in the appendix in order to avoid that this paper is dominated by too much colorful pictures.

3.2.1 About Linear Trends

As described in section 3.1 and section 3.1.3 white and brown noise processes were used as the origins for new time series that show a specific behavior. The motivation to take white noise was that it can be seen as an approximative model for financial returns series. Further on it can be used to produce brown noise which itself can be seen as an approximative model for financial logarithmic price series that have no trend.

Taking the white noise in figure 8 to produce the brown noise in figure 9 and adding a linear trend to it leads to figure 11. This can be formulated as follows:

\[ Y_n = B_n + m \cdot n \]

where \( B_n \) is brown noise as described in section 3.1.2. If we now derivate that process we get the time series in figure 12. Note that this time series is the same as the one in figure 8, just shifted upwards by \( m \). Therefore the wavelet spectras of figure 12 and figure 8 are identical. Actually if a time series is passed to the function that calculates the wavelet coefficients, it calculates first the mean of that time series. Then it subtracts the mean from the original time series and performs the wavelet transformation with that modified time series.

Let’s say we want to simulate business cycles. A time series like that would have an oscillating linear trend. If the differentiated process is analyzed it is not necessary to add a linear trend and an oscillating signal to the brown noise. The result is the same if the oscillating signal is added alone.
3.2.2 Reality Check

In figure 10 the exponential brown noise with linear trend is shown. The three plots on that page can now be compared to a corresponding real world financial time series. The SP500 was taken to do that comparison and visualized in figure 13, figure 14 and figure 15. The variance, mean and length of the artificial and the real world time series are very close.

From far away the two processes look somewhat similar. By taking a closer look clear differences can be seen, especially if the return series (figure 12 and figure 15) are compared. Since the power spectrum of a white noise process is $\chi^2$ distributed, 5% of the values are within the thick contour lines and those areas are randomly distributed within the spectrum. This can be seen in figure 8. In contrast the significant areas in figure 15 are clustered together which indicates that that process is less random.

On the one hand this shows that white noise can indeed be used as an approximation for financial time series. On the other hand it also shows that the process has to be further modified to get closer to real financial time series. The most promising approach that will be examined in the following are probably volatility clusters.

3.2.3 Brown Noise Modifications

Jumps, steps or oscillations could be observed within financial logarithmic price series. For this reason those features were added to linear brown noise (see figure 11) and can therefore be described as deviations from a linear trend. On the other hand the wavelet spectras of differentiated modified linear brown noise and differentiated modified plain brown noise are identical (see section 3.2.1).

Jumps and steps (section 3.3.1) can be described as structural breaks. Within the wavelet spectrum they lead to vertical significance levels. This can be explained by taking into consideration that the wavelet coefficients can be used to reconstruct the original time series (see section 2.4.6). So at the point in time where e.g. a jump is present it needs more "oscillations" to represent that anomaly.

Often seasonal cycles are present within stochastic time series. Those can be described as oscillations (section 3.3.2) and lead to horizontal significance levels. To understand this we can think of the wavelet coefficients $|W_n(s)|^2$ as a scalar product again (see section 2.3). So if the wavelet function lies on a part within the time series that has the same frequency as the wavelet function itself, the resulting value will be high.

3.2.4 White Noise Modifications

Volatility clusters can be observed within financial returns series. For this reason this feature was directly implemented into white noise (see figure 8) and can therefore be described as a deviation from a constant variance.
A volatility cluster (section 3.3.3) can be described as a structural break and similar to jumps they lead to vertical significance levels. The reason for this lies within the nature of the wavelet spectrum which is (at least for AR(1) processes) correlated to the variance (see section 2.4.3). Therefore the periods where the variance is higher will dominate the spectrum.

By just looking at the wavelet spectrum it can be hard to differentiate between spikes or volatility clusters within the time series. Therefore it always pays off to consult the original time series as well. E.g. by having a look at figure 15 it seems likely that the process is dominated by volatility clusters (see also section 4).
3.3 Examples

In this section every feature mentioned in section 3.2 will be visualized. Those are jumps and steps (section 3.3.1), oscillations (section 3.3.2) and volatility clusters (section 3.3.3). During the genesis of this paper much more examples evolved. But basically they are all variations of the three features visualized in this section and can be found in the appendix.

Figure 16 in section 3.3.1 shows the differentiated modified linear brown noise. The signal that was used can be seen in figure 18 and the derivation of it in figure 19 as well as in the top plot of figure 16 (red line). How plain brown noise (upper part of the plot) and linear brown noise (lower part of the plot) look like if the signal is added to them can be seen in figure 17. Note that both of those processes would lead to the same power spectrum in figure 16. The figures in section 3.3.2 and most of the examples in the appendix are done in the same spirit.

Figure 24 in section 3.3.3 shows the modified white noise that has volatility clusters in it. That noise was used to created plain and linear brown noise which is shown in figure 25 and figure 26.

All the wavelet spectras were calculated using $p = 0.95$ (see section 2.4.5) and $\delta_j = 0.25$ (see section 2.3.1).
3.3.1 Jumps and Steps

Figure 16: The derivation of the brownian motion with the signal and linear trend included.

Model
The signal can be best seen in figure 18. The first peak is a jump whose height is equal to 30% of the maximal value of the brownian motion. For the second peak it is 40% and for the third peak 50%. The fourth and fifth peaks are steps having a height of 50% as well. The sixth peak has 70%, the seventh 100% and the last one 120%. Let’s describe the signal as $X_n$ which is one for the jump positions and the points between the steps. Otherwise it shall be zero. Then the modified brown noise (figure 17) and the derivative of that noise (figure 16) can be described as follows:

$$Y_n = B_n + X_n + m \cdot n, \quad y_n = Y_n - Y_{n-1}$$

where $m \cdot n$ describes the linear trend which could also be omitted (see section 3.2.1).

Interpretation
In this setting a jump is actually a step up and immediately down again. Therefore the impact of a jump is stronger since two steps are merged. Note that first, the nature of the features didn’t change through differentiating and second, the signal-noise ratio became much better. See also section 3.2.3 for a general interpretation of jumps and steps and section 3.3 for more information about the plots.
Figure 17: Evolution of the time series.

Figure 18: The original signal.

Figure 19: The derivation of the signal.
3.3.2 Oscillations

Model
The signal (figure 22) was generated as follows:

\[ X_n = A \sin \left( \frac{2\pi \lambda n}{\lambda} \right) \]

where \( A \) is 10\% of the maximum value of the brownian motion and \( \lambda = 12 \). Then the modified brown noise (figure 21) and the derivative of that noise (figure 20) can be described as follows:

\[ Y_n = B_n + X_n + m \cdot n, \quad y_n = Y_n - Y_{n-1} \]

where \( m \cdot n \) describes the linear trend which could also be omitted (see section 3.2.1).

Interpretation
The wavelet spectrum in figure 20 shows a clear indication for \( \lambda = 12 \). As described in section 2.3.3 one can distinguish between a spike in the data and a harmonic component by comparing the width of a significant area within the power spectrum to the width of the COI. Note that first, the nature of the feature didn’t change through differentiating and second, the signal-noise ratio became much better. See also section 3.2.3 for a general interpretation of oscillations and section 3.3 for more information about the plots.
Figure 21: Evolution of the time series.

Figure 22: The original signal.

Figure 23: The derivation of the signal.
3.3.3 Volatility Clusters

![Figure 24: White noise with volatility clusters included.](image)

**Model**
The volatility clusters were directly included into the white noise by defining four regions where the standard deviation is higher than for the rest of the process. Those regions are indicated through the red line in figure 24. The base volatility is $\sigma = 0.05$. Within the first region the standard deviation is 2 times higher, within the second region 2.5 times, within the third region 4 times and within the last region 6 times. This process was directly visualized in figure 24. To get an idea how brown noise looks like if there are volatility clusters within the underlying process, plain brown noise (figure 25) and linear brown noise (figure 26) were created using the white noise of figure 24.

**Interpretation**
As expected the regions that have higher variance dominate the spectrum. In real world financial time series this can often be seen and usually corresponds to financial crises. The brown noise plots support the theory outlined in section 3.1.2 that it doesn’t make much sense to look directly at their spectrum. See also section 3.2.4 for a general interpretation of jumps and steps and section 3.3 for more information about the plots.
Figure 25: Brown noise generated from the white noise that has volatility clusters.

Figure 26: Brown noise generated from the white noise that has volatility clusters and additional linear trend.
4 Real World Time Series

This section focuses on examining long term real world financial time series that date back until the time before the great depression. By relying on section 2 and section 3 questions arise of whether we can detect structural breaks (like jumps, steps or volatility clusters) or seasonal cycles (like oscillations) within those time series. And if so it would be interesting to know if those findings can be assigned to real world events or if they appear randomly. Analysis concerning those questions were done in section 4.2 while section 4.1 presents some background information on those analysis.

4.1 Introduction

Section 4.1.1 describes the data source of the time series that were used for the examples in section 4.2. To assign areas within the spectrum to real world events the same source was used. Section 4.1.2 presents a model on how one can think about the power spectrum in a less mathematical way.

4.1.1 Data Source

The long term financial time series that were used in section 4.2 were first presented within a presentation hold on the 5th r/metrics workshop in 2011 on the meielisalp (lake thun) [4]. Those time series provide monthly index prices starting on the 1923-12-31 until the 2010-12-31. Also there was a figure in that presentation that assigns historical events to a turnpoints analysis. For every example in section 4.2 a similar figure was created by assigning historical events to the significant areas within the wavelet spectrum. The historical events mentioned in the meielisalp presentation served as a basis for those figures.

4.1.2 The Trader Model

The trader model is an idea found in the book of Gençay et al. [2]. Let’s have a look at the power spectrum of e.g. figure 27. Then one can think of a scale (or frequency) (see also section 2.3.2) as a class of traders that have a time horizon similar to that scale. Which means that impacts on the lower part of the spectrum can be assigned to the behavior of investors with a short time horizon (like p.e. high frequency traders) while impacts on the upper part of the spectrum can be assigned to the behavior of investors with a long time horizon (p.e. pension funds manager). While it can probably be argued on how close this model is to reality it offers in my opinion a very practical access to the spectrum of a financial time series. Furthermore it raises interesting questions. Namely on how the decision of a trader on a low scale is influencing the decision of a trader on a high scale and vice versa.
4.2 Examples

In the following, three index price series will be presented and analyzed. First, this is the "Standard & Poor’s 500" equity index (S&P 500). Records before 1957 were taken from the S&P 90 index. Second, US government bonds (USGOVT) will be examined. And third, the commodity index from the commodity research bureau (CRB) will be analyzed. Records before 1956 were taken from the commodity index of the bureau of labor statistics (BLS). More index series can be found in the appendix.

As outlined in section 3.1.2, directly analyzing the original or logarithmic price series is not a good approach. To illustrate this the wavelet spectras of the original and logarithmic price series will be shown in the following examples. But the analysis will be based solely on the wavelet spectrum of the returns series. The returns series were calculated as follows:

\[
r_t = \log(\frac{S_t}{S_{t-1}}) = \log(S_t) - \log(S_{t-1})
\]

Within the wavelet spectras of all the examples the significant areas (see also section 2.4.5) are clustered together which indicates that a process is less random. In fact those areas can probably always be assigned to a real world event like a financial or political crisis. During such times the volatility tends to increase which will be visible within the spectrum, as described in section 3.2.4 and visualized in section 3.3.3. Therefore a selection of financial and political events is shown within the figures of the returns series in order to explain the significant areas within the spectrum. Figure 27 in section 4.2.1 shows in the upper part the logarithmic price series together with the returns series. In the lower part the power spectrum (see section 2.4) of the returns series is shown. The figure is overlaid with various financial and political events. Figure 28 shows the wavelet spectrum of the original price index and figure 29 the spectrum of the logarithmic price series. Figure 30 shows again the spectrum of the returns series but without showing the logarithmic price series in the upper part of the plot and the historical events. The figures in section 4.2.2 and section 4.2.3 as well as the figures in the appendix are done in the same spirit.

All the wavelet spectras were calculated using \( p = 0.95 \) (see section 2.4.5) and \( \delta_j = 0.25 \) (see section 2.3.1).
4.2.1 Standard & Poor’s 500

Interpretation
First note that the wavelet spectras of the original price series (figure 28) and the logarithmic price series (figure 29) are not that meaningful. Whereas the power spectrum of the returns series (figure 30) shows very localized behavior. In the sense that a lot of the power is concentrated in very specific points in time.

In figure 27 the areas that have significantly higher power than the rest of the spectrum are compared to a selection of political and financial crises. Comparing them with each other shows a high accordance of significant areas with real world events.

As already mentioned in section 4.2 it seems that those significant areas are caused mainly by volatility clusters. It doesn’t seem that there are localized jumps or steps and a hint for seasonal cycles can not be seen. Besides that the points in time where the process is shocked do not seem to have a rule.

Remarkable in figure 27 is the amount of power concentrated around the great depression. If this event would be taken out of the analysis, the time around the dot-com bubble and the recent sub prime crises would probably be the most significant.
Figure 28: The original price series.

Figure 29: The logarithmic price series.

Figure 30: The returns series.
4.2.2 US Government Bonds

Interpretation
As in section 4.2.1 the wavelet spectras of figure 32 and figure 33 are not of much interest and the significant areas within the power spectrum of figure 31 are again in accordance to the shown political and financial crises. Also volatility clusters seem to be the dominant feature again and the points in time where the process is shocked do not seem to have a rule.

But for bonds the dominant shocks within the system are due to different historic events than the historic events that shocked the S&P 500 or the commodities. It is the banking crisis after the great depression in 1931 and the savings & loan crisis in 1979 that shocked the bonds the most. In contrary the savings & loan crisis doesn’t seem to have a strong impact on the S&P 500.

By comparing the returns series of the S&P 500 (figure 30) with the returns series of the bonds (figure 34) it seems that the bonds have percental more power in higher scales than the equities. If this is the case then this would make sense according to the trader model (see section 4.1.2) which states that the impact of traders with a longer time horizon is within higher scales.
Figure 32: The original price series.

Figure 33: The logarithmic price series.

Figure 34: The returns series.
4.2.3 Commodity Research Bureau

Interpretation
As in section 4.2.1 the wavelet spectras of figure 36 and figure 37 are not of much interest and the significant areas within the power spectrum of figure 35 are again in accordance to the shown political and financial crises. Also volatility clusters seem to be the dominant feature again and the points in time where the process is shocked do not seem to have a rule.
But for commodities the dominant shocks within the system are due to different historic events than the historic events that shocked the S&P 500 or the bonds. It does make sense that the oil crisis in 1973 and the recent burst of the food bubble in 2007 are clearly visible within the wavelet spectrum and have much more weight than they have within the wavelet spectras of the S&P 500 and the bonds.
Figure 36: The original price series.

Figure 37: The logarithmic price series.

Figure 38: The returns series.

42
5 Outlook

It has been shown that the wavelet analysis is a useful method for analyzing stochastic time series and offers interesting insights into the dynamics of such processes. The focus of this paper is on examining financial price series. Of course the wavelet analysis is also suited to examine other time series, like economic variables or the relationships between different price series. One could as well move away from finance and examine e.g. climatic time series like Torrence and Compo [6] did. Another idea would be to move into the field of medical data to e.g. study electrocardiograms.

While this paper is rather a qualitative approach in order to convey a better understanding about the nature of the wavelet spectrum, there are plenty of quantitative applications such as:

- Seasonality Filtering
- Denoising
- Fitting
- Forecasting
- Cross correlation analysis (multiscale)
- Outlier detection

In the paper "Wavelets in Economics and Finance: Past and Future" [5], Ramsey offers a broad summary of how wavelet analysis can be applied in economics and finance.

At least for financial data the wavelet spectrum usually indicates that the time series under consideration tends to be a highly non stationary random process. An approach to get a better grip on this behavior is to treat every vertical slice (scale) of the wavelet spectrum separately. This opens a wide field of applications and tailored solutions to a problem. The paper of Ramsey [5] offers an overview of studies that have chosen that approach.

Denoising can be seen as an alternative for smoothing. But in contrary to smoothing where the random process is "averaged", denoising "filters" the wavelet spectrum. If we assume that a financial process is a sum of a smooth signal and noise, then smoothing is the method of choice. But if the signal is not smooth itself, then denoising promises better results [5].

Despite it seems impossible to forecast random shocks there are some questions worth to consider about shocks within the wavelet spectrum. If fluctuations are constrained, can we model them by investigating each scale separately? How is a shock on one scale influencing other scales (see also the trader model in section 4.1.2)?

Finally the wavelet spectrum could be used to define a stability index similar to the ones that have been shown in the meielisalp presentation [4].

43
6 Appendix

6.1 More Artificial Time Series

At the moment the following examples are given without explanations. More information, e.g. the exact frequencies used in figure 43, can be found within the presentation mentioned in section 1.3.

6.1.1 Periodic Trend (cut tips)

Figure 39: The derivation of the brownian motion with the signal and linear trend included.
Figure 40: Evolution of the time series.

Figure 41: The original signal.

Figure 42: The derivation of the signal.
6.1.2 Periodic Trend (super position)

Figure 43: The derivation of the brownian motion with the signal and linear trend included.
Figure 44: Evolution of the time series.

Figure 45: The original signal.

Figure 46: The derivation of the signal.
6.1.3 Periodic Trend (frequency modulation)

Figure 47: The derivation of the brownian motion with the signal and linear trend included.
Figure 48: Evolution of the time series.

Figure 49: The original signal.

Figure 50: The derivation of the signal.
6.1.4 Periodic Trend (amplitude modulation)

Figure 51: The derivation of the brownian motion with the signal and linear trend included.
Figure 52: Evolution of the time series.

Figure 53: The original signal.

Figure 54: The derivation of the signal.
6.1.5 Jumps

Figure 55: The derivation of the brownian motion with the signal and linear trend included.
Figure 56: Evolution of the time series.

Figure 57: The original signal.

Figure 58: The derivation of the signal.
6.1.6 Combo Steps and Jumps

Figure 59: The derivation of the brownian motion with the signal and linear trend included.
Figure 60: Evolution of the time series.

Figure 61: The original signal.

Figure 62: The derivation of the signal.
6.1.7 Master Combo

Figure 63: The derivation of the brownian motion with the signal and linear trend included.
Figure 64: Evolution of the time series.

Figure 65: The original signal.

Figure 66: The derivation of the signal.
6.2 More Real World Time Series

At the moment the following examples are given without explanations.

6.2.1 Dow Jones Industrial Average

Figure 67: History.
Figure 68: The original price series.

Figure 69: The logarithmic price series.

Figure 70: The returns series.
6.2.2 Swiss Performance Index (starting 1968)

Figure 71: History.
Figure 72: The original price series.

Figure 73: The logarithmic price series.

Figure 74: The returns series.
References


