Chapter 5

The Term Structure of Interest Rates

```
> library(fBasics)
> library(empfin)
> data(LiborRates)
> library(YieldCurve)
> data(FedYieldCurve, package = "YieldCurve")
> data(ECBYieldCurve, package = "YieldCurve")
> data(LiborRates, package = "empfin")
```

The relationship between yield and maturity is called the "term structure of interest rate." Since there are many ways to measure yield, there are also many types of term structures. The term structure of interest rates is important because, everything else being equal, maturity is the main determinant of yield. In addition, investors and economists believe that the shape of this curve may reflect the market's expectation of future rates.

5.1 Term Structure Shapes

The term structure of interest rates can assume a variety of shapes, as illustrated by the following 2 examples.
Figure 5.1 displays a sample of yield curves, published by the European Central Bank (ECB, 2013). These curves represent the yield to maturity for AAA rated Euro-zone government bonds. We show quarterly data from December 2006 to June 2009. Three patterns are visible:

- an almost flat curve (August 2007);
- an inverted curve at the short end, and upward sloping at the long end (January 2008);
- an evenly upward-sloping curve (December 2006, June 2009).

The data is extracted from the `YieldCurve` package, data set `ECBYieldCurve`:

```
> tau <- c(3/12, 6/12, 1:30)
> d1 <- as.Date("2006-12-29")
> d2 <- as.Date("2009-07-24")
> nbObs <- dim(ECBYieldCurve)[1]
> dtObs <- seq(d1, d2, length.out = nbObs)
> indx <- seq(1, nbObs, 90)
> YC <- ECBYieldCurve[indx, ]
> dtObs <- dtObs[indx]
> nbSample <- length(indx)
```

The plot is generated by:

```
> mat = dtObs[1] + (tau * 365)
> plot(mat, YC[1, ], type = "l", ylim = c(0, 5), ylab = "Euro Sovereign Bond Yield",
     col = 1, lwd = 2)
> for (i in seq(2, nbSample)) {
     mat = dtObs[i] + (tau * 365)
     lines(mat, YC[i, ], type = "l", col = i, lwd = 2)
 }
> legend("bottomright", legend = as.character(dtObs), col = seq(1,
     nbSample), lty = 1, lwd = 2)
```

Figure 5.2 (page 36) presents a similar plot for the US Treasury yield, over the period 1982-2009. This illustrates the wide range of slope that a yield curve can assume over a long period of time. The plot uses the data set `FedYieldCurve`, and is obtained as follows:

```
> tau <- c(3/12, 6/12, 1, 2, 3, 5, 7, 10)
> dtObs <- index(FedYieldCurve)
> nbObs <- dim(FedYieldCurve)[1]
> indx <- seq(1, nbObs, 12)
> YC <- FedYieldCurve[indx, ]
> dtObs <- dtObs[indx]
> nbSample <- length(indx)
> mat = dtObs[1] + (tau * 365)
> plot(mat, YC[1, ], type = "l", ylim = c(0, 15), xlim = c(dtObs[1],
     dtObs[nbSample] + 10 * 365), col = 1, lwd = 2, xlab = "Maturity",
     ylab = "Treasury Yield")
> for (i in seq(2, nbSample)) {
     mat = dtObs[i] + (tau * 365)
     lines(mat, YC[i, ], type = "l", col = i, lwd = 2)
 }
```

5.2 BUILDING A ZERO-COUPON BOND CURVE

Principle

In a stylized manner, the calculation of a zero-coupon bond yield curve is straightforward. Assume a set of bonds with cash flows $F_{i,j}$ occurring at regular intervals, where i is the index of the bond and j the index of the

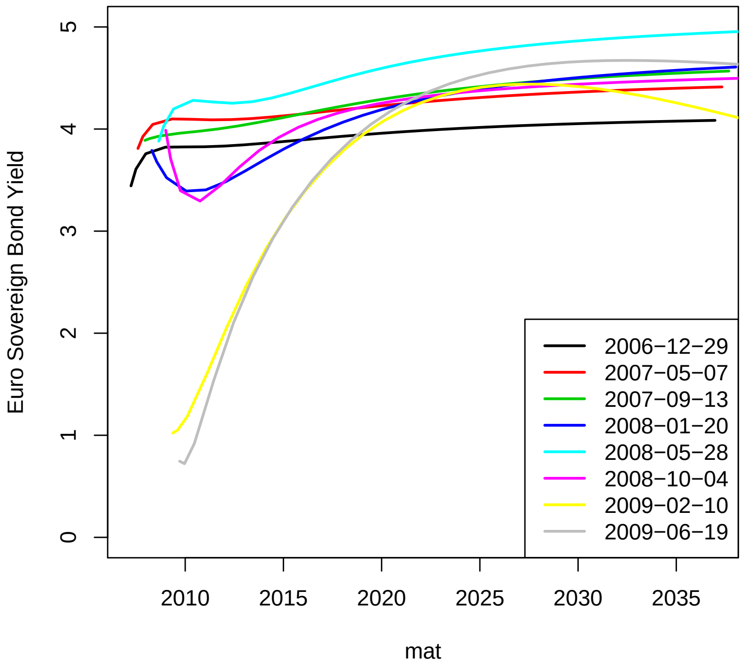

FIGURE 5.1: The yield to maturity curves of Euro-zone AAA government bonds

cash flow date. We look for an interest rate function $z(t)$ that prices the bonds exactly:

$$P_i = \sum_{j=1}^{n_i} F_{i,j} e^{-z(t_j)t_j}$$

The function $z(t)$ is called the continuously compounded zero-coupon yield curve, or "continuous zero curve" in short.
How to perform the calculation is best explained with an example. Assume that we observe four bonds, as summarized in Table 5.1.
Let y_i be the bond yield for bond i maturing in i years, and P_i the corresponding present value.

```
> y <- c(5.5, 6, 7.5, 8.5)/100
> c <- c(9, 7, 5, 4)/100
> P <- sapply(seq(4), function(i) SimpleBondPrice(c[i], i, y[i]))
> z <- rep(0, 4)
> r <- rep(0, 4)
```

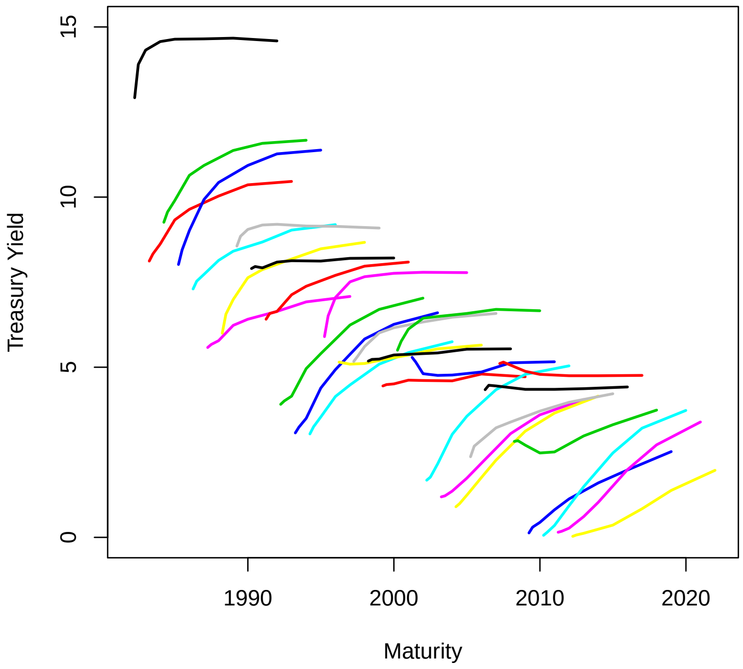

FIGURE 5.2: The yield to maturity curves of US government bonds

Maturity	Coupon (%)	Bond yield (%)
1	9	5.5
2	7	6.0
3	5	7.5
4	4	8.5

Table 5.1: Bond data for calculating a US Treasury zero-coupon curve

The zero-coupon rates z_i for each maturity are computed recursively, one maturity at a time. The one-year zero-coupon rate is obtained by converting the bond yield into a continuously compounded rate, using the definition of the continuous rate:

$$e^{-z_1} = \frac{1}{(1+y_1)}$$

or,

$$z_1 = \ln(1+y_1)$$

which gives $z_1 = 5.35\%$.
The zero coupon rate for year i, with the rates up to year $i-1$ assumed known, is obtained by solving for z_i:

$$P_i = 100\left(\sum_{j=1}^{i-1} ce^{-jz_j} + (1+c)e^{-iz_i}\right)$$

which yields:

$$z_i = -\frac{1}{i}\log\left(\frac{1}{1+c}\left(\frac{P_i}{100} - c\sum_{j=1}^{i-1} e^{-jz_j}\right)\right)$$

This recursive calculation is performed by the function `ZCRate`:

```
> ZCRate <- function(c, z, i) {
      zc <- -(1/i) * log(1/(1 + c[i]) * (P[i]/100 - c[i] * sum(exp(-seq(i -
          1) * z[1:(i - 1)]))))
      zc
  }
> for (i in seq(2, 4)) {
      z[i] <- ZCRate(c, z, i)
  }
```

A third term structure of interest is the par yield curve. For a given maturity, the par yield is the coupon rate of a bond priced at par (i.e. priced at 100%). Formally, given a zero-coupon curve, the par yield r_i for maturity i is such that:

$$1 = r_i \sum_{j=1}^{i} e^{-jz_j} + e^{-iz_i}$$

Back to our example, the par yields are computed by the following function, noting that the one-year par rate is by definition the corresponding bond yield:

```
> ParRate <- function(i) {
      r <- (1 - exp(-i * z[i]))/sum(exp(-seq(i) * z[1:i]))
      r
  }
> r[1] <- y[1]
> for (i in seq(2, 4)) {
      r[i] <- ParRate(i)
  }
```

Figure 5.3 displays the bond yield curve and the corresponding zero-coupon curve and par curves.
In practice, this simple method is not realistic for several reasons:

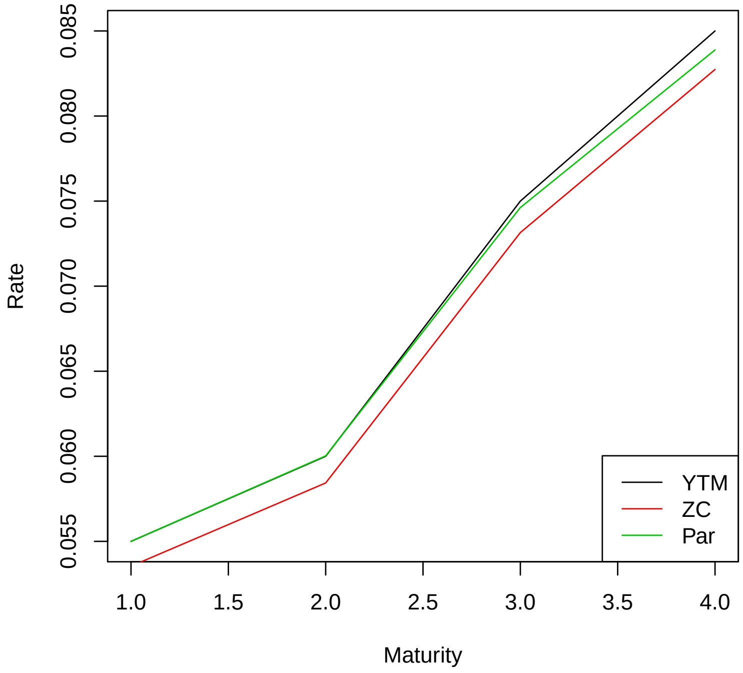

FIGURE 5.3: The yield to maturity (YTM) curve and the corresponding zero-coupon (ZC) and par (Par) curves

- in most markets, we will not find a set a bonds with identical anniversary dates;

- the method outlined above only provides values of $z(t)$ at maturity dates corresponding to the cash flow dates. An interpolation method is needed to obtain $z(t)$ at arbitrary maturities; and finally

- it does not account for the fact that some bonds may trade at a premium or discount: for example, a bond with a small outstanding amount will often trade at a discount, while the current 10-year benchmark bond will trade at a premium.

If we insist on a zero-coupon curve that prices exactly every bond in the sample, we will get a unnatural shape for the zero curve, and even more so for the forward rates.

The alternative is to specify a functional form (sufficiently flexible) for the zero-coupon curve, and to estimate its parameters by minimizing the pricing error over a set of bonds.
Since the term structure of interest rate can equivalently be represented in terms of spot rates, forward rates or discount factors, we have three choices for specifying the functional form of the term structure.

Functional forms for the zero-coupon curve

Modeling the instantaneous forward rate A popular method is to model the instantaneous forward rate, $f(t)$. Integrating this function gives the zero-coupon rate. Nelson and Siegel (1987) introduced the following model for the instantaneous forward rate:

$$f(t)=\beta_0+\beta_1 e^{-\frac{t}{\tau_1}}+\beta_2\frac{t}{\tau_1}e^{-\frac{t}{\tau_1}}$$

Integrating $f(t)$, the corresponding zero-coupon rates are given by:

$$z(t)=\beta_0+\beta_1\frac{1-e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}}+\beta_2\left(\frac{1-e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}}-e^{-\frac{t}{\tau_1}}\right)$$

Svensson (1994) extended this model with two additional parameters, defining the instantaneous forward rate as:

$$f(t)=\beta_0+\beta_1 e^{-\frac{t}{\tau_1}}+\beta_2\frac{t}{\tau_1}e^{-\frac{t}{\tau_1}}+\beta_3\frac{t}{\tau_2}e^{-\frac{t}{\tau_2}}$$

The following script, taken from the `YieldCurve` package, illustrates the fitting of the Nelson-Siegel and Svensson models to US Treasury yields.

```
> tau <- c(3, 6, 12, 24, 36, 60, 84, 120)
> Parameters.NS <- Nelson.Siegel(rate = FedYieldCurve[5, ], maturity = tau)
> Parameters.S <- Svensson(rate = FedYieldCurve[5, ], maturity = tau)
> tau.sim <- seq(1, 120, 2)
> y.NS <- NSrates(Parameters.NS, maturity = tau.sim)
> y.S <- Srates(Parameters.S, maturity = tau.sim, whichRate = "Spot")
```

Figure 5.4 shows a comparison of the two fitted spot curves.
The corresponding instantaneous fitted forward curves are displayed in Figure 5.5.

```
> f.S <- Srates(Parameters.S, tau.sim, whichRate = "Forward")
> forward.NS <- function(params, tau) {
      beta_0 <- as.numeric(params$beta_0)
      beta_1 <- as.numeric(params$beta_1)
      beta_2 <- as.numeric(params$beta_2)
      lambda <- as.numeric(params$lambda)
      MoverTau <- lambda * tau
      eMoT <- exp(-MoverTau)
      beta_0 * rep(1, length(tau)) + beta_1 * eMoT + beta_2 * MoverTau *
          eMoT
```

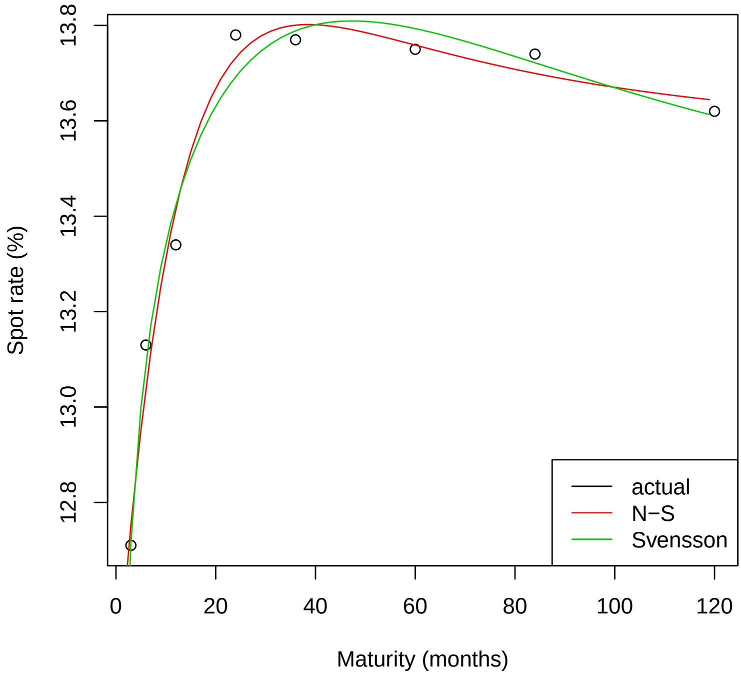

FIGURE 5.4: Fitted US Treasury zero-coupon spot curves, with Svensson and Nelson-Siegel's models

```
  }
> f.NS <- forward.NS(Parameters.NS, tau.sim)
```

The Svensson or Nelson-Siegel curves can be easily decomposed into three elementary shapes that capture the main features of a yield curve: level, slope and curvature. Figure 5.6 (page 42) shows the decomposition of the Svensson fitted spot curve into these three components.

Empirical justification The functional forms chosen by Svensson or Nelson and Siegel are supported by empirical observations. Indeed, a principal components analysis of daily changes in zero-coupon rates shows that actual changes are accounted for by three factors that have shapes consistent with the decomposition illustrated in Figure 5.6.

The data set consists of AAA Euro-zone government rates, observed from December 29, 2006 to July 24, 2009.

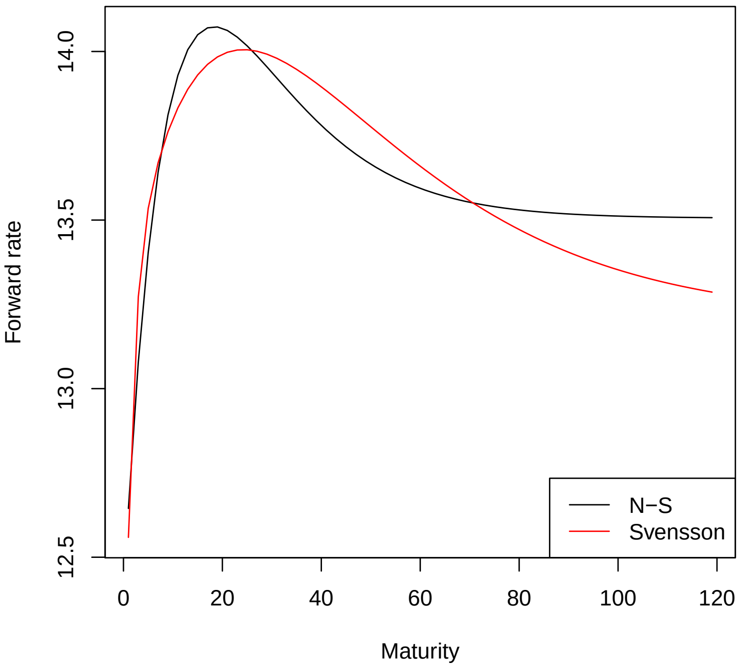

FIGURE 5.5: Fitted Treasury Zero-coupon forward curves, with Svensson model and Nelson-Siegel model

```
> n <- 252
> daily.zc.change <- diff(ECBYieldCurve)[2:(n + 1), ]
> pca <- princomp(daily.zc.change, center = TRUE, scale = TRUE)
```

Figure 5.7 shows the contribution of the first six factors to the total variance, and shows that of the bulk of the variance (about 90%) is captured by the first three factors.

The factor loadings show the sensitivity of zero-coupon of various maturities to factor movements. The first three factors are displayed in Figure 5.8 (page 44).

Focusing on the shapes for maturities over two years, the first factor (red) can be interpreted as a level factor, since it affects all zero-coupons in a similar way (except for the short maturities). The second factor (blue) can be interpreted as a slope or rotation factor. Finally, the third factor (green) can be interpreted as a curvature factor. We find a good qualitative agreement with the decomposition of Figure 5.6.

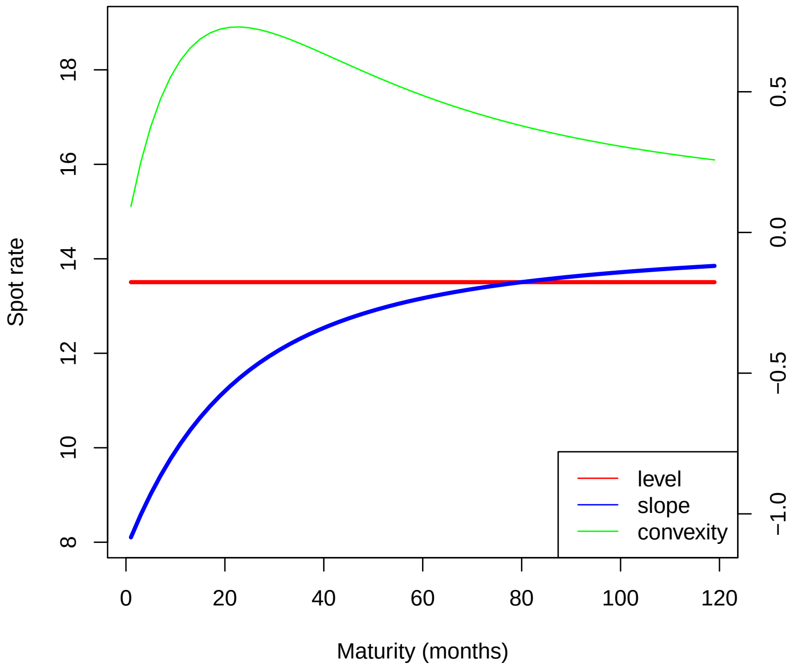

FIGURE 5.6: Components of fitted US Treasury zero-coupon spot curves, with the Nelson-Siegel model

5.3 BOOTSTRAPPING A ZERO-COUPON LIBOR CURVE

Given a set of Libor deposit rates and swap rates, we would like to compute the zero-coupon curve z_i. Again, this is best explained by an example. We use the `LiborRates` data set, which contains a time series of deposit rates and swap rates for various maturities. The construction of this data set was described in Chapter 1. The data for November 10, 2009, is displayed in Table 5.2.

Bootstrap from the deposit rates

The first part of the calculation is simple, and amounts to converting the Libor deposit rates ($\frac{\text{ACT}}{360}$) into zero-coupon continuous rates ($\frac{\text{ACT}}{365}$), using the relation:

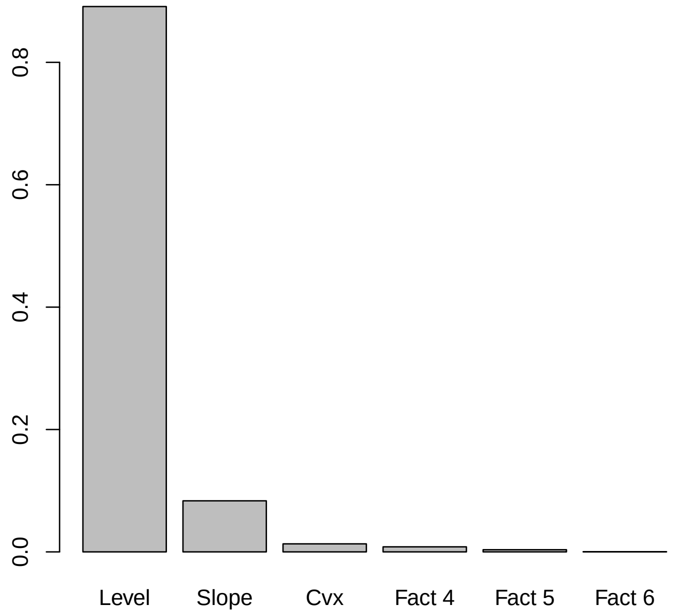

FIGURE 5.7: Contribution (in %) of the first six principal components (factors) to total variance (norm of the covariance matrix of changes in zero-coupon rates).

```
> dtToday <- as.timeDate("2009-11-10")
> l <- LiborRates[dtToday]/100
> print(l)
GMT
           Libor1M Libor2M Libor3M Libor6M Libor12M Swap1Y Swap2Y Swap3Y Swap4Y
2009-11-10  0.0024  0.0025  0.0027  0.0054   0.0112 0.0052 0.0117 0.0178 0.0225
           Swap5Y Swap7Y Swap10Y Swap30Y
2009-11-10 0.0262 0.0315  0.0359  0.0423
```

Table 5.2: One row of the LiborRates data set.

$$(1+\frac{d}{360}l)=e^{z\frac{d}{365}}$$

where:

$d =$ actual number of days from settlement date to maturity

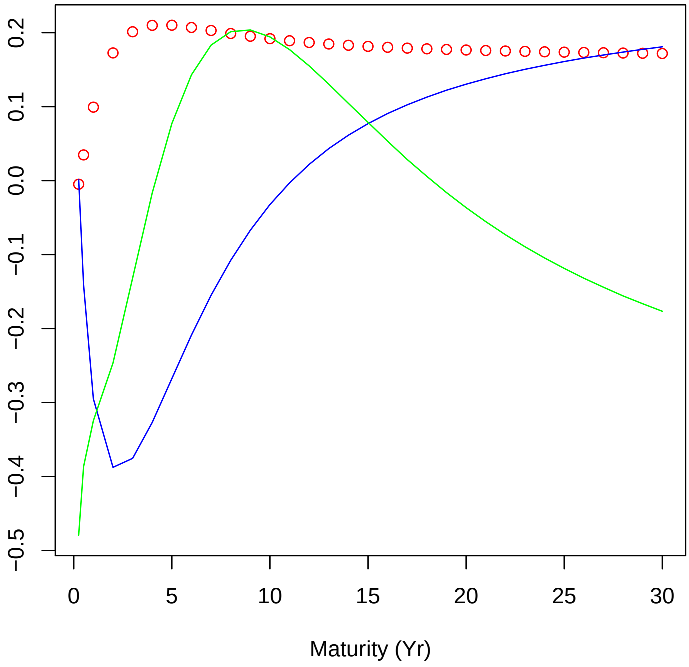

Figure 5.8: Factor loadings for first three principal components.

$l =$ deposit rate

$z =$ zero-coupon rate

The following statements convert the five deposit rates into zero-coupon continuous rates:

```
> dtToday <- as.timeDate("2009-11-10")
> dtPayment.dep <- timeCalendar(d = c(10, 10, 10, 10, 10), m = c(12,
      1, 2, 5, 11), y = c(2009, rep(2010, 4)))
> z.dep = rep(0, 5)
> mat.dep = as.numeric(dtPayment.dep - dtToday)
> for (i in seq_along(z.dep)) {
      dd = mat.dep[i]
      r <- as.numeric(l[1, i])
      z.dep[i] = log(1 + r * dd/360) * 365/dd
  }
```

Bootstrap from the swap rates

At one year and beyond, the calculation proceeds recursively, one year at a time. The swap rate is a par yield, meaning that the fixed leg of a swap has a PV of 100. We can use the same logic as in the previous section, and compute recursively the zero-coupon rate for each maturity.
The calculation is performed by the following script, where we interpolate the swap coupon rates for the missing maturities.
The interpolation is performed with:

```
> dtExpiry.swap <- timeSequence(from = as.timeDate("2010-11-10"),
     by = "year", length.out = 30)
> libor.mat <- c(2, 3, 4, 5, 7, 10, 30)
> nbRates <- length(l)
> libor.coupon <- l[7:nbRates]
> f <- splinefun(libor.mat, libor.coupon, method = "monoH.FC")
> mat.swap <- as.numeric(dtExpiry.swap - dtToday)/365
> cp.swap <- sapply(mat.swap, f)
```

The recursive computation of the zero-coupon rates is finally done with:

```
> z.libor <- rep(0, length(mat.swap))
> P <- rep(100, length(mat.swap))
> z.libor[1] <- z.dep[5]
> for (i in seq(2, length(z.libor))) {
     z.libor[i] <- ZCRate(cp.swap, z.libor, i)
 }
```

The resulting zero-curve is the concatenation of the zero-coupon deposit rates and the zero-coupon swap rates. It is shown in Figure 5.9 (page 46).

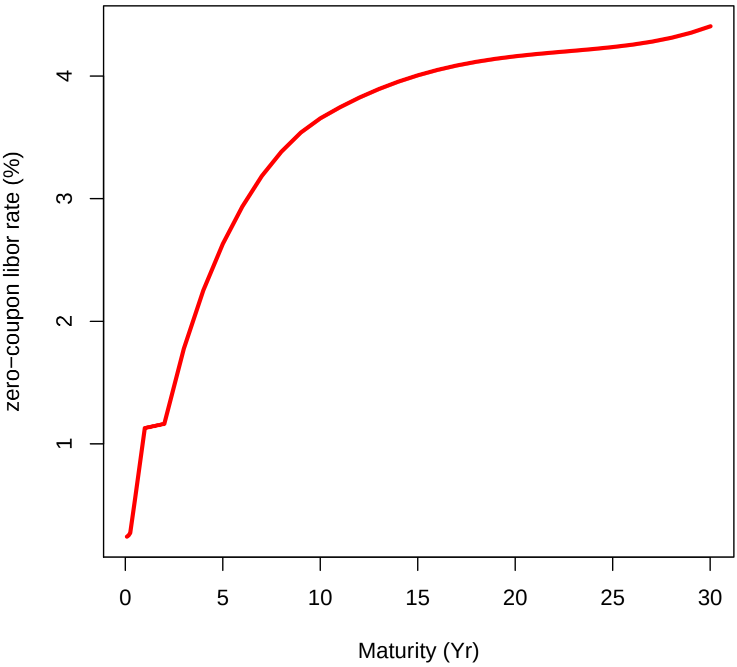

Figure 5.9: Zero-coupon curve bootstrapped from Libor deposits and swaps rates.