Chapter 8

Laplace Inversion

Geman and Yor [1993] computed an expression for the Laplace transform of the Asian option price. Thus numerical inversion remains to obtain the price. Unfortunately the result has a quite complicated structure from which one has to determine the final price by a nontrivial Laplace inversion. Geman and Eydeland [1995] attempted to invert the Geman-Yor expression numerically using a fast Fourier transform. Fu, Madan, and Wang [1998] and Craddock, Heath, and Platen [2000] used numerical Laplace inversion methods on the same expression, although both report on problems for short maturities and/or small volatilities. In addition Sudler [1999] valuated Asian options making use of the relationship to Fourier transforms.

8.1 Geman and Yor's Formalism

According to Geman and Yor [1993] the price of an Asian Call option can be written as

$$e^{-rT}\mathbb{E}[(\mathscr{A}_T - X)^+] = e^{-rT}\left(\frac{4S}{\sigma^2 T}\right) C^{(\nu)}(h,q)\,, \tag{8.1}$$

where, in the notation of Geman and Yor,

$$\nu = \frac{2r}{\sigma^2} - 1\,,\;\; h = \frac{\sigma^2 T}{4}\,,\;\; q = \frac{\sigma^2 T}{4}\frac{X}{S}\,. \tag{8.2}$$

The Laplace transform of $C^{(\nu)}(h,q)$ in the first parameter h is given by

$$\hat{C}(\lambda,q) = \int_0^\infty e^{-\lambda h} C^{(\nu)}(h,q)\,dh = \frac{\int_0^{1/(2q)} e^{-x} x^{(\mu-\nu)/2-2}(1-2qx)^{(\mu+\nu)/2+1}\,dx}{\lambda(\lambda-2-2\nu)\Gamma((\mu-\nu)/2-1)}\,, \tag{8.3}$$

with $\mu = \sqrt{2\lambda+\nu^2}$.

8.2 NUMERICAL LAPLACE INVERSION

Geman and Yor [1993] notice, that the inversion of this Laplace transform for a fixed h is not easy, since the parameter λ also appears in the Gamma function. They remark that "there is no of-the-shelf software for inverting Laplace transforms, which may be useful for a numerical solution of this problem." Public available software prior to 1993 included the TOMS 619 package of Piessens and Huysens [1984], the TOMS 662 package of Garbow, Giunta, Lyness and Murli [1988], and the TOMS 682 package by Murli and Rizzardi [1990]. Fu, Madan and Wang [1997] also claim that a naive attempt at inverting this transform may lead to erroneous results. They employed the methods of Euler and Post-Widder, as proposed by Abate and Whitt [1995].

A more recent Laplace inversion method, going back to D'Amore, Laccetti, and Murli [1999a, 1999b], seems to us better suited for solving the problems with the Laplace inversion. Their method is based on a Fourier series expansion of the inverse transform. They also analyzed the corresponding discretization error and demonstrated how the expansion can be used in the development of an "automatic routine", one in which the user needs to specify only the required accuracy. This is a big advantage in comparison to the Euler and Post-Widder implementations, where several convergence parameters have to be specified.

8.3 EQUIVALENCE TO THE FOURIER TRANSFORM

In the spirit of this approach, and following Sudler [1999], we express the Laplace transform of $C^{(\nu)}(h,q)$ by Kummer's confluent hypergeometric function $M(a,b,z)$, i.e.

$$C^{(\nu)}(\lambda,q) = \frac{(\frac{1}{2q})^a}{\lambda(\lambda-2-2\nu)} \frac{\Gamma(b-a)}{\Gamma(b)} M\left(\frac{1}{2}(\mu-\nu),\mu,-\frac{1}{2q}\right) . \tag{8.4}$$

Note, that the indexes of the Kummer function M are functions of λ. In order to make the function regular in the upper-half complex plane, we introduce the function $\widetilde{f}(h,q) = e^{-\alpha h} f(h,q)$, where $\alpha = 2(1+\nu)/\sigma^2$. Then $\widetilde{C}(\lambda,q) = C(\alpha + i\lambda, q)$ and the inverse Fourier Transform is given by

$$\widetilde{f}(h,q) = \frac{1}{2pi} \int_{-\infty}^{\infty} e^{i\lambda h} \widetilde{C}(i\lambda,q) d\lambda . \tag{8.5}$$

Hence we finally arrive at

$$C^{\nu}(h,q) = e^{\alpha h} \widetilde{f}(h,q) , \tag{8.6}$$

which allows us to derive the call price via equation (8.1).

8.4 COMPLEX GAMMA AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

For the evaluation of the Asian option the "Gamma function" $\Gamma(z)$ with complex argument z and the "log-Gamma function" $\ln\Gamma(z)$ for "extreme" values are needed. For an implementation one has to take care to the complicated branch cut structure inherited from the logarithm function. In addition the confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$ with complex argument z and indexes a and b are required. These are degenerate forms of the hypergeometric function which arise as a solution of the "confluent hypergeometric differential equation" $xy''+(c-x)y'-ay=0$. The functions are also called Kummer's function $M(a,b,z)$ of the first and $U(a,b,z)$ of the second kind. These functions are given by

$$
\begin{aligned}
M(a,b,z) &= \frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)z^n}{\Gamma(b+n)n!}\,, \\
U(a,b,z) &= \frac{\pi}{sin(\pi b)}\left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right].
\end{aligned}
\tag{8.7}
$$

Since the complex Gamma and Kummer functions are not available in R's base package we have implemented them from Fortran routines. It's worth to note, that Kummer's functions were implemented from the TOMS 707 algorithm of Nardin, Berger and Bhalla [1992]. Their Fortran routines are suited for complex arguments and complex indexes with large magnitudes. In the case of small volatilities and/short times to maturity it is very important to prevent the calculations from numerical overflows and thus the Kummer functions must also support the direct calculation of their logarithms. The algorithm makes use of a direct summation of the Kummer series. As further sources to confluent hypergeometric functions and their relationships to other complex functions, we refer to the textbooks of Abramowitz and Stegun [1972], and Slater [1960], as well as to Weinstein's [2003] MathWorld web resource.
Before we valuate Asian options by Laplace inversion, we take a closer look at the function which has to be integrated. For low volatilities and/or short times to maturity the function values get extremely small and oscillates with a very slow decaying slope. Note, that the real parts of the indexes of Kummer's function $\mathscr{R}(a)$ and $\mathscr{R}(b)$ can take values larger than 1000, which mean that $M(a,b,z)$ takes on extremely large values. These facts limit the evaluation of Asian call prices to values $\sigma^2 T \gtrsim 0.002$.
Table 4 lists the results for Asian call prices obtained from our numerical implementation in comparison with some results we found in the literature.

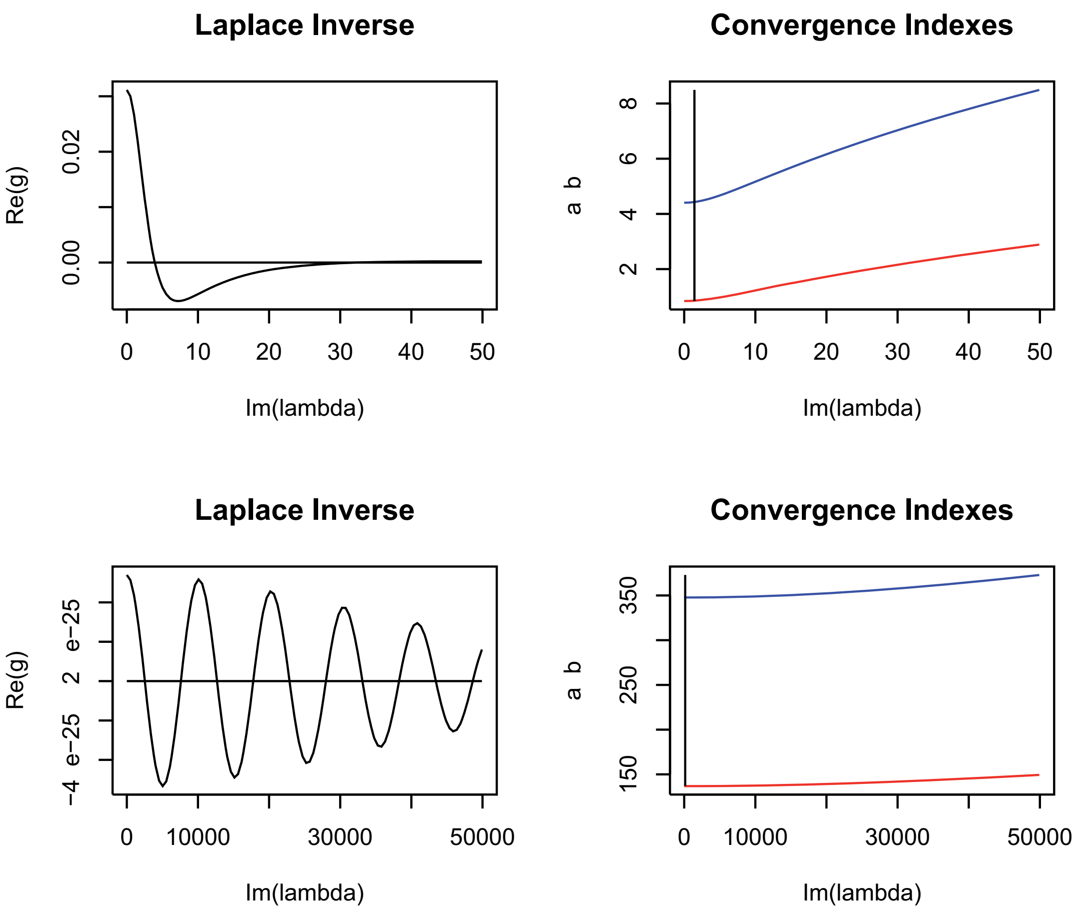

FIGURE 8.1:
The figures to the left show the Real part of the function g to be Laplace inverted for two settings, "high volatility - long maturity" and "low volatility - short maturity", i.e. $\sigma^2 T = 0.25$ and 0.0025, respectively. The figures to the right display the "big numbers" for the real part of the two indexes a and b, appearing in Kummer's confluent hypergeometric function.

X S=100	T σ=0.02	Euler FMW	CPU Sparc-10	PostWidder FMW	CPU Sparc-10	FT our Results	CPU Pentium4
90	0.4	11.5293	53	11.5176	640	11.5249	14
95		7.2131	44	7.1981	631	7.2106	11
100		3.8087	42	3.8196	628	3.8073	11
105		1.6465	45	1.6623	623	1.6459	11
110		0.5761	38	0.5728	625	0.5759	11
90	0.5	11.9247	34	11.9241	613	11.9251	10
95		7.7249	33	7.7185	591	7.7252	9
100		4.3696	33	4.3759	601	4.3698	9
105		2.1175	32	2.1290	585	2.1176	8
110		0.8734	32	0.8753	595	0.8735	9
90	1.0	13.8372	26	13.8439	518	13.8315 *	5
95		9.9998	26	10.0029	513	9.9957 *	6
100		6.7801	26	6.7823	514	6.7773 *	5
105		4.2982	25	4.3010	503	4.2965 *	5
110		2.5473	25	2.5450	503	2.5462 *	5
90	2.0	17.1212	24	17.1297	475	17.0988	3
95		13.6763	23	13.6830	473	13.6584	4
100		10.6319	23	10.6370	473	10.6180	4
105		8.0436	23	8.0474	469	8.0331	3
110		5.9267	22	5.9295	474	5.9189	4
90	3.0	19.8398	21	19.8495	468	19.7960	3
95		16.6740	21	16.6822	467	16.6372	4
100		13.7974	21	13.8042	467	13.7669	3
105		11.2447	20	11.2502	462	11.2199	3
110		9.0316	20	9.0360	465	9.0116	4
90	5.0	24.0861	19	24.0861	453	23.9780	3
95		21.3774	22	21.3774	442	21.2814	4
100		18.8399	19	18.8399	449	18.7553	3
105		16.4917	21	16.4917	441	16.4176	4
110		14.3442	21	14.3442	453	14.2798	3

FIGURE 8.2:
Comparison of numerical techniques for pricing continuous Asian options by Laplace inversion and direct Fourier transformation: The table summarizes results from the Euler and Post-Widder approach with settings recommended by Abate and Whitt as calculated by Fu, Madan and Wang [1997] and our results obtained by direct Fourier transformation. Note, for $T = 1$ our results market by the star are identical to those presend by Zhang []. The values obtained by the Euler and Post-Whitter approach overestimate for all applied parameters the call prices, in same cases significantly.

8.5 R-FUNCTIONS

The function `GemanYorAsianOption()`

The function `GemanYorAsianOption()` valuates Asian options from the Laplace transform via the Fourier transform and `GemanYorFun()` calculates the function itself to be transformed. For the Fourier Transform the R-function `integrate()` from R's base package is used, which allows for an adaptive quadrature of functions. This function uses routines written by Piessens and deDoncker-Kapenga [1983].

The function `Gamma()`

The function `Gamma()` allows to calculate the complex Gamma function.

The functions `kummerM()` and `kummerM()`

The functions `kummerM()` and RfunkummerU allow to calculate the Kummer functions of the first and second kind with complex argument and complex indexes.