Chapter 3

Statistical Series Expansions

We can systematically improve an approximation to the state price density by a series expansion of the "true" density $f(x)$ around a fitted or approximative density $a(x)$

$$f(x) = a(x) + \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=1}^{N-1} (-1)^j \frac{[\kappa_j(f) - \kappa_j(g)]}{j!} \frac{d^j}{dx^j} \right]^i a(x) + \varepsilon(x) \qquad (3.1)$$

where the $\kappa_j(\cdot)$ are the cumulants of the distributions $f(x)$ and $a(x)$. Specific orderings of the terms and special choices of the density function $a(x)$ lead to several different final forms of this series expansion known under the names Gram-Charlier and Edgeworth expansions. Jurczenko, Maillet and Negrea [2002] gave an overview on these approaches.

3.1 Gram-Charlier Series Expansion

Successive derivatives of $a(x)$ up to the fourth order yield

$$f(x) = a(x) + P_1(k) \frac{d\,a(x)}{dx} + P_2(k) \frac{d^2 a(x)}{dx^2} + P_3(k) \frac{d^3 a(x)}{dx^3} + P_4(k) \frac{d^4 a(x)}{dx^4} + \ldots, \qquad (3.2)$$

where the Polynomials $P_n(k)$ are defined as

$$\begin{aligned} P_1(k) &= k_1\,, \qquad (3.3) \\ P_2(k) &= \frac{1}{2!}[k_2 + k_1^2]\,, \\ P_3(k) &= \frac{1}{3!}[k_3 + 3k_2 k_1 + 3k_1^3]\,, \\ P_4(k) &= \frac{1}{4!}[k_4 + 4k_3 k_1 + 3k_2^2 + 6k_2 k_1^2 + k_1^4]\,, \end{aligned}$$

and $k_i = \kappa_i(f) - \kappa_i(g)$. This type of ordering and collection of terms is called *Gram-Charlier* series expansion.

3.2 EDGEWORTH SERIES EXPANSION

If x in equation ... is a normalized sum of *iid* variables one can set up a proper asymptotic series expansion. The type of ordering is based on the fact that for a sum of *iid* standardized random variables the j-th cumulant is proportional to $n^{1-j/2}$, with $j \geq 2$. Developing and collecting terms of equal order in $n^{-1/2}$ in equation (3.1), say up to order n^{-1}, $f(x)$ can then be expressed as

$$f(x) = a(x) - n^{-1/2} \frac{k_{i,3}}{3!} \frac{d^3 a(x)}{dx^3} + n^{-1} \left[\frac{k_{i,4}}{4!} \frac{d^4 a(x)}{dx^4} + 10 \frac{k_{i,3}^2}{6!} \frac{d^6 a(x)}{dx^6} \right] + \dots, \tag{3.4}$$

where $k_{i,j} = \kappa_{i,j}(f) - \kappa_{i,j}(a)$. $\kappa_{i,j}$ is the j-th cumulant of the standardized random variable $\sigma^{-1}(x_i - 1)$, $k_{i,1} = 0$ and $k_{i,2} = 1$ with $(1 \times j) = (1, \dots, n) \times (3, 4)$. This grouping of terms is known as *Edgeworth* series expansion. Second and third terms in equation (3.3) allow to adjust $a(x)$ according to the gap between the skewness and the kurtosis of the density.
Here we consider the log-Normal, the reciprocal-Gamma and the Johnson-Type-I as approximative distributions $a(x)$ and perform Gram-Charlier series expansions around these distributions. An investigation on these three distributions can also be found in the work of Datey, Gauthier, and Simonate [2002]. The case of the log-Normal distribution goes back to Jarrow and Rudd [1982].

Gram-Charlier log-Normal Model

We assume that the approximative state price density is the log-Normal distribution $\ell(x)$. Applying a Gram-Charlier series expansion, the state price density reads

$$f(x) \approx \ell(x) - \frac{m_3^{(f)} - m_3^{(\ell)}}{3!} \frac{d^3 \ell(x)}{dx^3} + \frac{m_4^{(f)} - m_4^{(\ell)}}{4!} \frac{d^4 \ell(x)}{dx^4}, \tag{3.5}$$

where $m_n^{(f,\ell)}$ are the central moments of the “true” Asian and the “approximative” log-Normal density respectively. With the help of the identity, used by Jarrow and Rudd,

$$\int_{-\infty}^{\infty} (x - X) \frac{d^j a(x)}{dx^j} dx = \left[\frac{d^{j-2} a(x)}{dx^{j-2}} \right]_{x=X}, \tag{3.6}$$

valid for $j \geq 2$, we can calculate the call price of an Asian option as

$$c^{GCLN} = c^{LTW} - e^{-rT}\left[\frac{\kappa_3}{3!}\frac{d\ell(x)}{dx} - \frac{\kappa_4}{4!}\frac{d^2\ell(x)}{dx^2}\right]_{x=X}. \tag{3.7}$$

The first term c^{LTW} counts for the Levy-Turnbull-Wakeman Asian call price, given by equation (2.4), with the first two Asian density moments matched to the log-Normal density. The next two terms take care for the corrections due to skewness and kurtosis effects which are not fully captured by the log-Normal distribution.

Gram Charlier Reciprocal-Gamma Model

In the same way we can develop the Gram-Charlier expansion for the reciprocal-Gamma density. In expression (3.5) we replace $\ell(x)$ with the reciprocal-Gamma distribution $g^R_{\alpha,\beta}(x)$ as defined in (2.6) and calculate again by partial integration the Asian call price

$$c^{GCRG} = c^{MP} - e^{-rT}\left[\frac{\kappa_3}{3!}\frac{d\, g^R_{\alpha,\beta}(x)}{dx} - \frac{\kappa_4}{4!}\frac{d^2 g^R_{\alpha,\beta}(x)}{dx^2}\right]_{x=X}. \tag{3.8}$$

The first term c^{MP} is the call price derived by Milevsky and Posner from the reciprocal-Gamma distribution as given in equation (2.7) and the remaining third and fourth order terms correct for remaining skewness and kurtosis effects.

Gram-Charlier Johnson-Type-I Model

Since for the moment matched Johnson Type I distribution the first three moments are in agreement between the “true” and the “approximated”, the third order term vanishes and we are left with the fourth order term. Again density and call price can be derived, yielding

$$c^{GCJI} = c^{PM} + e^{-rT}\left[\frac{\kappa_4}{4!}\frac{d^2 z(x)}{dx^2}\right]_{x=X}, \tag{3.9}$$

where c^{PM} counts for the call price derived by Posner and Milevsky based on the 3-moment matched Johnson Type I distribution. The additional term corrects for remaining kurtosis effects.

In all three cases the derivatives can be computed in a straightforward way, but lead to some lengthy expressions and therefore they will be not listed here. The expressions can be found in the implemented R functions for the option pricing formulae.

X S=100	r	σ T=1	RST-LB	GC-LN	GC-RG	GC-JI	T-UB
95	0.05	0.05	7.1777	7.1777	7.1777	7.1777	7.1778
100			2.7162	2.7161	2.7162	2.7162	2.7162
105			0.3372	0.3373	0.3372	0.3373	0.3374
95	0.09		8.8088	8.8088	8.8088	8.8088	8.8089
100			4.3082	4.3082	4.3082	4.3082	4.3084
105			0.9583	0.9584	0.9584	0.9584	0.9585
95	0.15		11.0941	11.0941	11.0941	11.0941	11.0941
100			6.7944	6.7944	6.7943	6.7944	6.7945
105			2.7444	2.7444	2.7445	2.7444	2.7446
90	0.05	0.10	11.9511	11.9513	11.9509	11.9511	11.9522
100			3.6413	3.6409	3.6419	3.6413	3.6416
110			0.3311	0.3315	0.3309	0.3312	0.3322
90	0.09		13.3852	13.3854	13.3850	13.3852	13.3860
100			4.9151	4.9146	4.9158	4.9150	4.9154
110			0.6301	0.6307	0.6298	0.6303	0.6310
90	0.15		15.3988	15.3988	15.3987	15.3988	15.3992
100			7.0277	7.0275	7.0281	7.0277	7.0285
110			1.4133	1.4138	1.4132	1.4137	1.4143
90	0.05	0.20	12.5956	12.5881	12.6066	12.5949	12.6007
100			5.7627	5.7585	5.7676	5.7624	5.7644
110			1.9892	1.9965	1.9818	1.9908	1.9927
90	0.09		13.8312	13.8257	13.8395	13.8307	13.8372
100			6.7770	6.7703	6.7856	6.7763	6.7787
110			2.5455	2.5515	2.5389	2.5469	2.5485
90	0.15		15.6416	15.6394	15.6449	15.6415	15.6489
100			8.4085	8.3996	8.4212	8.4075	8.4104
110			3.5547	3.5575	3.5518	3.5559	3.5578
90	0.05	0.30	13.9524	13.8956	14.0354	13.9453	13.9620
100			7.9444	7.9295	7.9625	7.9428	7.9505
110			4.0701	4.0991	4.0349	4.0752	4.0786
90	0.09		14.9828	14.9254	15.0691	14.9756	14.9928
100			8.8276	8.8019	8.8617	8.8244	8.8333
110			4.6949	4.7174	4.6662	4.6991	4.7027
90	0.15		16.5120	16.4586	16.5954	16.5054	16.5237
100			10.2087	10.1690	10.2667	10.2036	10.2141
110			5.7282	5.7377	5.7150	5.7306	5.7355
			Inside Bounds:	15	6	21	
			Outside Bounds:	21	30	15	

FIGURE 3.1:
This figure shows the results for Asian call prices obtained from Gram-Charlier series expansions around the log-Normal, GC-LN, the reciprocal-Gamma, GC-RG, and the Johnson-Type-I, GC-JI, distributions. The option parameters are the same as used in Table 1. We achieve a significant improvement in the precision of the prices, especially for low volatilities and/or short maturities. Much more prices are falling inside the lower and upper bounds, or at least they are moving towards the bounds.
For the log-Normal Gram-Charlier case the prices are in agreement with those calculated by Fusai and Tagliani [2002]. They show also prices calculated from a Gram-Charlier series expansion around the Normal distribution.

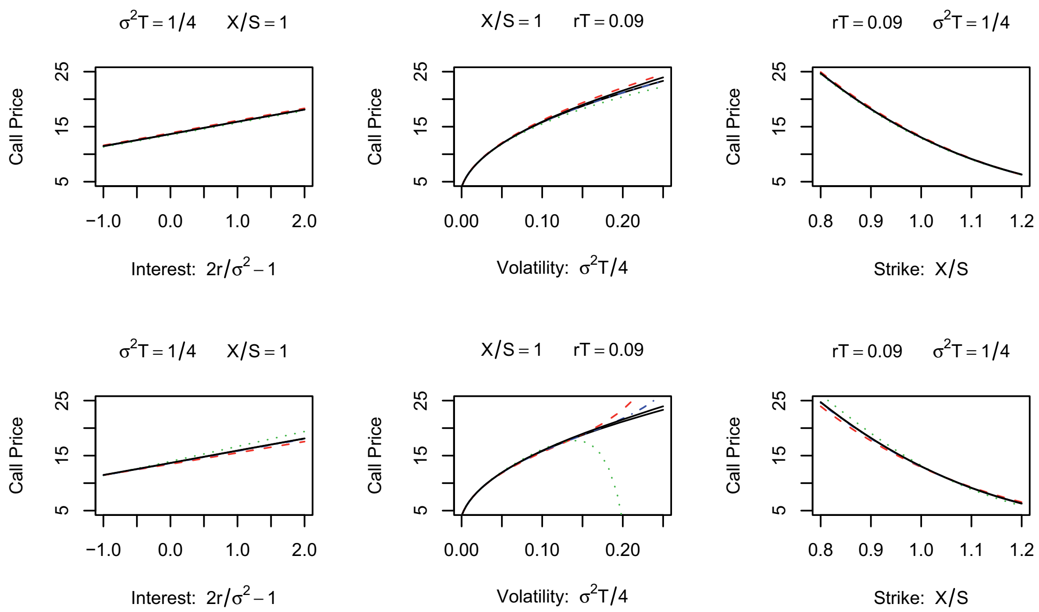

FIGURE 3.2:
The graphs show the call price deviations of the moment matched approximations improved by the Gram-Charlier series expansion. Prices are shown for the log-Normal GC-LN (dashed), the reciprocal-Gamma GC-RG (dotted), the Johnson-Type-I GC-JI (dashdotted) approaches, together with the lower, RST-LB (solid), and upper bounds, T-UB (solid), for various option parameter settings.
The first row of the graphs shows for at-the-money options the increasing call price deviations as a function of normalized interest rates $2r/\sigma^2 - 1$ for low, intermediate and high volatility scenarios. The graphs in the middle row show the variations of the call price deviations for fixed interest rate and high, intermediate and low volatility scenarios as a function of normalized moneyness X/S. Note, that prices for options which are high in-the-money are not very precisely evaluated by all three approximations. The graphs in the lower row show the deviations as function of the normalized volatility $\sigma^2 T/4$ for a fixed interest rate $rT = 0.09$ and options which are in-the-money ($X/S = 1.2$), at-the-money and out-of-the-money (0.8).
Here, Johnson Type I fits the option prices best, log-Normal and reciprocal-Gamma behave in most of the situations contrary, when one approximation overestimates the price, then the other one underestimates it, and vice versa.

3.3 R-FUNCTIONS

The function `GramCharlierAsianOption()`

The function `GramCharlierAsianOption()` allows to calculate Asian option prices from the Gram-Charlier series expansions for the log-Normal, the reciprocal-Gamma and the Johnson-Type-I distributions. The distributions can be selected by choosing one element from the functions argument `method=c("LN", "RG", "JI")`.